

A Vector Measure Approach to the Optional Stochastic Integral

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Introduction

In this paper, we shall present a method of defining the stochastic integral of a real-valued, optional process H with respect to certain Banach-valued processes X . We shall do so by defining an L_1 -valued measure J_X on the optional σ -algebra \mathcal{O} . This measure will be an extension of a measure I_X defined on the algebra of predictable rectangles (cf. [3], [9]).

Many constructions of stochastic integrals are limited by their reliance on the predictable σ -algebra. Consequently, only predictable processes may be integrated. By defining a measure directly on \mathcal{O} , we overcome this problem and are able to integrate optional processes. Other extensions of stochastic integrals to optional processes have relied on the geometry of Hilbert space and have been restricted to processes X which reduce to either square integrable martingales or martingales of integrable variation. We again overcome these restrictions and require only that X be a separably-valued, adapted, cadlag process with a certain uniform integrability condition satisfied by both types of martingales above. For convenience, we also assume $X_\infty = \lim_{t \rightarrow \infty} X_t$ exists a.s.

In a previous paper [4], we constructed an optional stochastic integral for Hilbert and/or real-valued processes H and X . For this construction, we sought a natural definition of the integral $H \circ X$, for bounded, optional H and square integrable X , in terms of a predictable stochastic integral. To this end, we let H' be a bounded, predictable process such that $H - H'$ is thin and $H'_0 = H_0$. Then, $H \circ X = H' \cdot X + Z$, where Z is a certain limit in M^2 of square integrable martingales. If $H = 1_{[S, T[}$, where S, T are stopping times with $S \leq T$, then we may take $H' = 1_{]S, T]} + 1_{(S=0 < T)}$. We then obtain a formula, given in the next section, which defines a finitely additive measure J_X on the algebra \mathcal{T} generated by the stochastic intervals $[S, T[$. We see that this formula does not rely on the square integrability of

X nor on the geometry of Hilbert space.

We can then define a stochastic integral $\int HdX$, for the previously described Banach-valued X , when $H = 1_{[S, T[}$. This integral will exist as a Banach-valued, adapted, cadlag process. When J_X has a countably additive extension to O , we call X O -summable. In this case, we can define $\int HdX$ for a class of real optional processes which includes all bounded optional processes.

In this paper, we shall prove the existence of $\int HdX$, when X is O -summable, as an adapted, cadlag process (Theorem 1.1). We shall show (Theorem 1.2) that $\int HdX$ is a (uniformly integrable) martingale when X is a (uniformly integrable) martingale. We shall see (Theorem 2.1) that when X is a Hilbert-valued H^1 - martingale, then X is O -summable. In particular, when X is a Hilbert-valued square integrable martingale, then for bounded H , $\int HdX$ agrees with the ordinary compensated stochastic integral $H \circ X$.

§1. Construction of the Optional Stochastic Integral

Let (Ω, \mathcal{F}, P) be a probability space having a filtration $(\mathcal{F}_t)_{t \geq 0}$ which satisfies the usual conditions. Then the optional σ -algebra O is generated by the family of stochastic intervals $\mathfrak{S} = \{[S, T[: S, T \text{ stopping times, } S \leq T\}$. We let \mathcal{T} be the algebra of sets of the form $B = [S_1, T_1[: \cup \cdots \cup [S_n, T_n[$, where S_i, T_i are stopping times and $S_1 \leq T_1 \leq \cdots \leq S_n \leq T_n$. Then O is also generated by \mathcal{T} .

Let G be a Banach space with the Radon-Nikodym property. Assume X is an adapted, cadlag, G -valued process with a separable range such that $X_\infty = \lim_{t \rightarrow \infty} X_t$ exists a.s. and the set $\{X_T : T \text{ stopping time}\} \cup \{X_{T-} : T \text{ stopping time}\}$ is uniformly (Bochner) integrable with respect to P .

For every stopping time T , we define a separably-valued process of integrable variation ${}^T A$ by ${}^T A_r = \Delta : X_T 1_{(T > 0)} 1_{(T \leq r)}$. We denote by $({}^T A)^P$ the predictable compensator of ${}^T A$, which exists by [7].

For $[S, T[: \epsilon : \mathfrak{S}$, we define $J_X([S, T[)$ by

$$J_X([S, T[) = X_T 1_{(T > 0)} - X_S 1_{(S > 0)} - [({}^T A)_\infty - ({}^T A)_\infty^P] + [({}^S A)_\infty - ({}^S A)_\infty^P]. \quad (1)$$

We then obtain a finitely additive measure $J_X : \mathcal{T} \rightarrow L_1(\Omega, \mathcal{F}, P, G)$ given by $J_X(B) = \sum_{i=1}^n J_X([S_i, T_i[)$, where $B = [S_1, T_1[: \cup \cdots \cup [S_n, T_n[$.

To see that J_X is finitely additive, we first note that if $[S, T[, [U, V[: \epsilon : \mathfrak{S}$ and $[S, T[: \cap [U, V[= \emptyset$, then

$$J_X([S, T[) = J_X([S \wedge U, T \wedge U]) + J_X([S \vee V, T \vee V]). \quad (2)$$

This is immediate since, by disjointness of the stochastic intervals,

$$\begin{aligned} X_T 1_{(T>0)} - X_S 1_{(S>0)} &= X_{T \wedge U} 1_{(T \wedge U > 0)} - X_{S \wedge U} 1_{(S \wedge U > 0)} \\ &\quad + X_{T \vee V} 1_{(T \vee V > 0)} - X_{S \vee V} 1_{(S \vee V > 0)}. \end{aligned} \quad (3)$$

Likewise,

$${}^T A - : {}^S A = : {}^{T \wedge U} A - : {}^{S \wedge U} A + : {}^{T \vee V} A - : {}^{S \vee V} A. \quad (4)$$

After subtracting predictable compensators and adding (3), equation (2) follows.

Next if $B = [S_1, T_1[: \cup \cdots \cup [S_n, T_n[: \epsilon: \mathcal{T}$ and $C = [U, V[: \epsilon: \mathcal{S}$ with $B \cap C = \emptyset$, then $B \cup C: \epsilon: \mathcal{T}$ and

$$\begin{aligned} B \cup C &= [S_1 \wedge U, T_1 \wedge U[\cup \cdots \cup [S_n \wedge U, T_n \wedge U[\cup [U, V[\\ &\quad \cup [S_1 \vee V, T_1 \vee V[\cup \cdots \cup [S_n \vee V, T_n \vee V[. \end{aligned} \quad (5)$$

Hence, by (2),

$$\begin{aligned} J_X(B \cup C) &= \sum_{i=1}^n J_X([S_i \wedge U, T_i \wedge U]) + J_X([U, V]) \\ &\quad + \sum_{i=1}^n J_X([S_i \vee V, T_i \vee V]) \\ &= \sum_{i=1}^n J_X([S_i, T_i]) + J_X([U, V]) \\ &= J_X(B) + J_X(C). \end{aligned} \quad (6)$$

Finally, we assume inductively that $J_X(B \cup C) = J_X(B) + J_X(C)$, where $B = [S_1, T_1[\cup \cdots \cup [S_n, T_n[$, n arbitrary, and $C = [U_1, V_1[\cup \cdots \cup [U_{m-1}, V_{m-1}[$ with $B \cap C = \emptyset$. Then let $D = C \cup [U_m, V_m[: \epsilon: \mathcal{T}$ with $B \cap D = \emptyset$. As in (5), $B \cup [U_m, V_m[$ can be written as an element in \mathcal{T} ; hence, by the induction hypothesis on C and (6),

$$\begin{aligned} J_X(B \cup D) &= J_X(B \cup [U_m, V_m]) + J_X(C) \\ &= J_X(B) + J_X([U_m, V_m]) + J_X(C) \\ &= J_X(B) + J_X(D). \end{aligned}$$

We say the process X is *O-summable* if J_X has a countably additive extension to the optional σ -algebra $O = \sigma(\mathcal{T})$. We shall always denote the extension again by J_X . The corresponding space of real-valued, O -measurable functions defined on $R^+ \times \Omega$, which are integrable with respect

to J_X , in the sense of Bartle-Dunford-Schwartz, will be denoted by $\mathcal{L}_1(J_X)$. Then all bounded, optional processes are elements of $\mathcal{L}_1(J_X)$.

We note that we can in fact define a Banach space $L_1(J_X)$ consisting of equivalence classes of integrable functions in $\mathcal{L}_1(J_X)$ (cf. [1]). The norm on this space is defined as follows. To simplify notation, we let $m = J_X$ and $B = L_1(\Omega, \mathcal{F}, P, G)$. Then m is a B -valued countably additive measure. We denote the unit sphere in the dual of B by B_1^* . For every $z: \epsilon: B_1^*$, we define $m_z: O \rightarrow \mathbf{R}$ by $m_z(A) = z(m(A))$ and we denote the variation of m_z by $|m_z|$. Then for $H \in \mathcal{L}_1(m)$,

$$\|H\|_{L_1(m)} = \sup \left\{ \int : |H| : d : |m_z| : : z \in B_1^* \right\}.$$

Although we shall not need to distinguish between $H: \epsilon: \mathcal{L}_1(m)$ and its equivalence class in $L_1(m)$, we shall make use of the following inequality:

$$\begin{aligned} E \left[\left\| \int H dJ_X \right\| G \right] &= \left\| \int H dJ_X \right\| B \\ &\leq \|H\|_{L_1(m)}. \end{aligned}$$

In the same manner that we defined J_X , we can define a measure J_X^t on the optional subsets O_t of $[0, t] \times \Omega$, generated by the family $\mathcal{S}_t = \{[S, T] : S \leq T \leq t\}$. Then J_X^t takes values in $L_1(\Omega, \mathcal{F}_t, P, G)$ for all $t \geq 0$. However, the restriction of J_X to O_t agrees with J_X^t on \mathcal{S}_t and thus will agree on O_t . It follows that if $H \in \mathcal{L}_1(J_X)$, then $H1_{[0, t]}: \epsilon: \mathcal{L}_1(J_X^t)$ and

$$\int H1_{[0, t]} d: J_X = \int H1_{[0, t]} d: J_X^t \epsilon L_1(\Omega, \mathcal{F}_t, P, G).$$

If there exists an adapted, cadlag process $Z: R^+ \times \Omega \rightarrow G$ such that, for all $t \geq 0$, Z_t is a member of the equivalence class $\int H1_{[0, t]} dJ_X$, then Z is called the *stochastic integral* of H with respect to X and is denoted by $H \circ X$ or $\int H dX$.

It follows that $\int H dX$ is unique when defined. For if Z_t and Y_t are both elements of the equivalence class $\int H1_{[0, t]} dJ_X$, then $Z_t = Y_t$ a.s. for each t . By right continuity, Z and Y are indistinguishable. Furthermore, the map $H \rightarrow \int H dX$ is linear.

Theorem 1.1. *If X is O -summable and $H \in \mathcal{L}_1(J_X)$, then the stochastic integral $\int H dX$ exists.*

Before we prove Theorem 1, we shall prove the following lemma and discuss the relation of this stochastic integral to the predictable stochastic integral.

Lemma 1.1. *Assume X is O -summable.*

- (a) *For every stopping time T , $J_X([T]) = 1_{(T=0)} X_0 + ({}^T A)_\infty - ({}^T A)_\infty^P$.*

- (b) For all stopping times S, T with $S \leq T$, $J_X(]S, T]) = X_T - X_S$.
(c) For all predictable rectangles $R_0 = \{0\} \times E$, with $E \in \mathcal{F}_0$, and $R =]s, t] \times F$, with $F \in \mathcal{F}_s$, $J_X(R_0) = 1_E X_0$ and $J_X(R) = 1_F(X_t - X_s)$.

Proof. (a) By countable additivity, $J_X([T]) = \lim_{n \rightarrow \infty} J_X([T, T + \frac{1}{n}])$. We then note that $X_{T + \frac{1}{n}} 1_{(T + \frac{1}{n}) > 0} = X_{T + \frac{1}{n}}$ converges in L_1 to X_T and $X_T - X_T 1_{(T > 0)} = X_T 1_{(T = 0)}$. Next $(^{T + \frac{1}{n}}A)_\infty = (X_{T + \frac{1}{n}} - X_{(T + \frac{1}{n})-})$ converges in L_1 to $X_T - X_T = 0$. Lastly, if we let $B^n = ^{T + \frac{1}{n}}A$, then

$$\begin{aligned} E[|(B^n)_\infty^P|] &\leq E[\int_0^\infty |d(B^n)_s^P|] \leq E[\int_0^\infty |dB_s^n|] \\ &= E[|\Delta X_{T + \frac{1}{n}}|] \xrightarrow{n} 0. \end{aligned}$$

Hence, $(^{T + \frac{1}{n}}A)_\infty^P$ also converges to 0 in L_1 . The result in (a) then follows from the definition of $J_X([T, T + \frac{1}{n}])$.

- (b) Part (b) follows from (a) since $J_X(]S, T]) = J_X([S, T]) - J_X([S]) + J_X([T])$.

- (c) If we let T be the stopping time 0_E , which is 0 on E and $+\infty$ on E^c , then $R_0 = [T]$. Then $^T A \equiv 0$ and hence $(^T A)^P \equiv 0$. Thus, $J_X(R_0) = J_X([T]) = 1_{(T=0)} X_0 = 1_E X_0$.

The set R is equal to the stochastic interval $]s_F, t_F]$; hence, by (b), $J_X(R) = X_{t_F} - X_{s_F} = 1_F(X_t - X_s)$. \square

We see that when X is O -summable, J_X agrees with the measure I_X , defined by Brooks and Dinculeanu in [3], on the algebra of predictable rectangles which generates the predictable σ -algebra \mathcal{P} . Thus, the extension of J_X to O will be an extension of I_X to \mathcal{P} ; hence, X is summable in the ordinary sense of the predictable stochastic integral. It is known by [10], that if X is a real-valued process, then X is summable if and only if X is a quasimartingale bounded in L_1 . If X takes values in a Banach space which does not contain a copy of c_0 , then X is summable if and only if I_X is bounded on the algebra of predictable rectangles [3]. Of course, when X is O -summable, then all other results concerning summable processes are still valid. In particular, if X is a G -valued summable quasimartingale and H is a real-valued, bounded, predictable process, then $\int H dX$ is a quasimartingale [3].

Proof of Theorem 1. We first assume that $H = 1_{[S, T[}$, where $[S, T[\in \mathcal{E}$. Then, for a fixed $t \geq 0$,

$$\begin{aligned} H 1_{[0, t]} &= 1_{]S, T]} 1_{[0, t]} - 1_{[T]} 1_{[0, t]} + 1_{[S]} 1_{[0, t]} \\ &= 1_{]S \wedge t, T \wedge t]} - 1_{[T_F]} + 1_{[S_E]}, \end{aligned}$$

where $F = \{w : T(w) \leq t\}$ and $E = \{w : S(w) \leq t\}$. Thus, by Lemma 1,

$$\begin{aligned} \int H 1_{[0, t]} dJ_X &= X_{T \wedge t} - X_{S \wedge t} - 1_{(T_F=0)} X_0 + 1_{(S_E=0)} X_0 \\ &\quad - ((^{T_F}A)_\infty - (^{T_F}A)_\infty^P) + (^{S_E}A)_\infty - (^{S_E}A)_\infty^P. \end{aligned}$$

However, ${}^{T_F}A_r = \Delta X_{T_F} 1_{(T_F > 0)} 1_{(T_F \leq r)} = \Delta X_T 1_{(T > 0)} 1_{(T \leq t)} 1_{(T \leq r)} = ({}^T A)_{t \wedge r}$. Thus, ${}^{T_F}A$ is simply the process ${}^T A$ stopped at t . Since stopping and compensation commute, $({}^{T_F}A)_r^P = ({}^T A)_{t \wedge r}^P$. Hence,

$$\int H 1_{[0,t]} dJ_X = X_{T \wedge t} - X_{S \wedge t} - 1_{(T=0)} X_0 + 1_{(S=0)} X_0 - (({}^T A)_t - ({}^T A)_t^P) + ({}^S A)_t - ({}^S A)_t^P. \quad (*)$$

Since X is an adapted, G -valued, cadlag process, we can take $(\int H dX)_t$ to be the process given in (*). Thus, $\int H dX$ exists for $H = 1_{[S, T[}$. Moreover, if we replace the deterministic time t by an arbitrary stopping time U in the above argument, we see that $(\int H dX)_U = \int H 1_{[0, U]} dJ_X$ in $L_1(\Omega, \mathcal{F}, P, G)$.

It follows first that if $H = 1_A$, where $A \in \mathcal{T}$, and secondly if H is a simple \mathcal{T} -measurable function then $\int H dX$ exists and for every stopping time U , $(\int H dX)_U = \int H 1_{[0, U]} dJ_X$ in $L_1(\Omega, \mathcal{F}, P, G)$.

We now let $H \in \mathcal{L}_1(J_X)$. Since \mathcal{T} generates the optional σ -algebra \mathcal{O} , the set of simple \mathcal{T} -measurable functions are dense in $L_1(J_X)$. We then choose a sequence $\{H^n\}$ of such functions which converge to H in $L_1(J_X)$. By choosing an appropriate subsequence, we may assume $\|H^{n+1} - H^n\|_{L_1(J_X)} < 4^{-n}$, for all n .

We let $Z^n = \int : H^n dX$ for all n . We fix $t_0 > 0$ and define a sequence of stopping times $\{U_n\}$ by

$$U_n = \inf\{t : \| : Z_t^{n+1} - Z_t^n : \| > 2^{-n}\} \wedge t_0,$$

and let $G_n = \{w : U_n(w) < t_0\}$. Then for all n ,

$$\begin{aligned} E[: \| Z_{U_n}^{n+1} - Z_{U_n}^n : \|] &= E[: \int (H^{n+1} - H^n) 1_{[0, U_n]} dJ_X : \|_{L_1(G)} \\ &\leq \| : H^{n+1} - H^n : \|_{L_1(J_X)} \\ &< 4^{-n}. \end{aligned}$$

Since $G_n \subseteq \{ : \| : Z_{U_n}^{n+1} - Z_{U_n}^n : \| : > 2^{-n}\}$, by Tchebyshev's inequality,

$$\begin{aligned} P(G_n) &\leq P(\{ : \| : Z_{U_n}^{n+1} - Z_{U_n}^n : \| : > 2^{-n}\}) \\ &\leq 2^n : E[: \| : Z_{U_n}^{n+1} - Z_{U_n}^n : \| :] \\ &< 2^{-n}. \end{aligned}$$

Hence, by the Borel-Cantelli lemma, $P(G_n \text{ i.o.}) = 0$. Thus, for a.e. w , there exists an integer N_w such that $U_n(w) = t_0$ when $n \geq N_w$. That is, for all $t : \epsilon : [0, t_0]$, $\| : Z_t^{n+1}(w) - Z_t^n(w) : \| : < 2^{-n}$ if $n \geq N_w$. It follows that the sequence $\{Z_{(\cdot)}^n(w)\}$ converges uniformly on $[0, t_0]$.

We then let $Z_t(w) = \lim_{n \rightarrow \infty} Z_t^n(w)$. Then for a.e. w , $Z_{(\cdot)}(w)$ is the uniform limit on $[0, t_0]$ of a sequence of cadlag functions; hence, Z is cadlag outside of an evanescent set. Moreover, since Z^n is adapted for all n , Z is also adapted.

Finally, for each t , $Z_t^n = \int H^n 1_{[0,t]} dJ_X$ in $L_1(\Omega, \mathcal{F}_t, P, G)$ and $\{\int H^n 1_{[0,t]} dJ_X\}$ converges to $\int H 1_{[0,t]} dJ_X$ in $L_1(G)$. It follows that $Z_t = \int H 1_{[0,t]} dJ_X$ in $L_1(G)$; hence, $Z = \int HdX$. \square

Theorem 1.2. *Let X be O -summable and let $H \in \mathcal{L}_1(J_X)$. If X is a martingale, then $\int HdX$ is a martingale. Moreover, if X is a uniformly integrable martingale, then so is $\int HdX$ and $(\int HdX)_t = E(\int HdJ_X | \mathcal{F}_t)$ a.s. for each $t \leq \infty$.*

Proof. If $H = 1_{[S, T[}$, where $[S, T[: \epsilon : \mathcal{S}$, then by (*) in the proof of Theorem 1, $\int HdX$ is a martingale. Since the processes ${}^T A - ({}^T A)^P$ and ${}^S A - ({}^S A)^P$ are uniformly integrable martingales, $\int HdX$ will also be uniformly integrable when X is and then clearly $E(\int HdJ_X | \mathcal{F}_t) = (\int HdX)_t$ a.s.

Since the map $H \rightarrow \int HdX$ is linear, the theorem remains true if $H = 1_A$, where $A \in \mathcal{T}$, and then if H is a simple \mathcal{T} -measurable process.

If $H \in \mathcal{L}_1(J_X)$, we let $\{H^n\}$ be a sequence of simple \mathcal{T} -measurable processes which converge in $L_1(J_X)$ to H . Then for all $t \leq \infty$, $\int H^n 1_{[0,t]} dJ_X \xrightarrow{n} \int H 1_{[0,t]} dJ_X$ in $L_1(\Omega, \mathcal{F}_t, P, G)$; hence,

$$\left(\int H^n dX \right)_t \xrightarrow{n} \left(\int HdX \right)_t$$

in $L_1(\Omega, \mathcal{F}_t, P, G)$. Thus, if $s \leq t$ and $A \in \mathcal{F}_s$,

$$\begin{aligned} \int_A \left(\int HdX \right)_s dP &= \lim_n \int_A \left(\int H^n dX \right)_s dP \\ &= \lim_n \int_A \left(\int H^n dX \right)_t dP = \int_A \left(\int HdX \right)_t dP; \end{aligned}$$

hence, $\int HdX$ is a martingale. Moreover, if X is uniformly integrable, then for $t = \infty$, $\int H^n dJ_X \xrightarrow{n} \int HdJ_X$ in L_1 ; thus, $E(\int H^n dJ_X | \mathcal{F}_s) \xrightarrow{n} E(\int HdJ_X | \mathcal{F}_s)$ in L_1 for all $s \leq \infty$. It follows that $(\int HdX)_s = E(\int HdJ_X | \mathcal{F}_s)$ a.s. \square

§2. The Optional Integral in Hilbert Space

We adopt the same setting as the previous section; however, we now let G be a real, separable Hilbert space. We shall give an example of a process X that is O -summable. We note that, even for real-valued X , the result yields a wider class of processes than square integrable martingales for which we can define the optional stochastic integral. This class is the

space H_G^1 which consists of all uniformly integrable G -valued martingales X such that $E[X^*] = E[\sup_t |X_t|] < \infty$, or equivalently, $E[[X, X]_\infty^{\frac{1}{2}}] < \infty$.

Theorem 2.1. *If G is a real, separable Hilbert space and $X \in H_G^1$, then X is O -summable.*

Proof. Since G is weakly sequentially complete, J_X takes values in a weakly sequentially complete space $L_1(\Omega, \mathcal{F}, P, G)$. Thus J_X will have a countably additive extension to O if it is weakly countably additive and bounded on the algebra \mathcal{T} which generates O .

We let $Y_\infty \in L_\infty(G) = (L_1(G))^*$ and let Y be a cadlag version of the bounded martingale $E(Y_\infty | \mathcal{F}_t)$. Then $Y \in (BMO)_G^1 = (H_G^1)^*$. Then for all $[S, T] \in \mathcal{S}$,

$$\begin{aligned}
Y_\infty(J_X([S, T])) &= E[\langle Y_\infty, J_X([S, T]) \rangle] \\
&= E[\langle Y_\infty, 1_{(S=0 < T)} X_0 \rangle] \\
&\quad + E[\langle Y_\infty, X_T \rangle] - E[\langle Y_\infty, X_S \rangle] \\
&\quad - E[\langle Y_\infty, ({}^T A)_\infty - ({}^T A)_\infty^P \rangle] + E[\langle Y_\infty, ({}^S A)_\infty - ({}^S A)_\infty^P \rangle] \\
&= E[\langle Y_0, 1_{(S=0 < T)} X_0 \rangle] + E[\langle Y_T, X_T \rangle] - E[\langle Y_S, X_S \rangle] \\
&\quad - E\left[\int^{\infty} \Delta Y_r d({}^T A)_r\right] + E\left[\int^{\infty} \Delta Y_r d({}^S A)_r\right] \\
&= E[\langle Y_0, 1_{(S=0 < T)} X_0 \rangle] + E[[X, Y]_T] - E[[X, Y]_S] \\
&\quad - E[\langle \Delta Y_T, \Delta X_T 1_{(T>0)} \rangle] + E[\langle \Delta Y_S, \Delta X_S 1_{(S>0)} \rangle] \\
&= E[[X, Y]_T] - E[[X, Y]_S] \\
&\quad - E[\langle \Delta X_T, \Delta Y_T \rangle] + E[\langle \Delta X_S, \Delta Y_S \rangle] \\
&= E\left[\int^{\infty} 1_{[S, T]} d[X, Y]_r\right] - E\left[\int^{\infty} (1_{[T]} - 1_{[S]}) d[X, Y]_r\right] \\
&= E\left[\int^{\infty} 1_{[S, T]} d[X, Y]_r\right].
\end{aligned}$$

It follows that if $B \in \mathcal{T}$, then $Y_\infty(J_X(B)) = E[\int^{\infty} 1_B d[X, Y]_r]$; hence, J_X is weakly countably additive on \mathcal{T} . Since $X \in H_G^1$ and $Y \in (BMO)_G^1$,

$E[\int^{\infty} |d[X, Y]_r|] \leq \sqrt{2} \|Y\|_{BMO} \|X\|_{H^1} < \infty$; hence, J_X is also weakly bounded on \mathcal{T} . \square

Of course, if X is a G -valued square integrable martingale, then $X \in H_G^1$. In this case, we can define a stochastic integral $H \circ X$ for all real, optional processes H such that $E[\int^{\infty} H_r^2 d[X, X]_r] < \infty$. This integral is characterized as follows: the process $(H \circ X)_t = E((H \circ X)_\infty | \mathcal{F}_t)$ is the unique G -valued square integrable martingale such that for every bounded G -valued martingale Y , $E[\langle Y_\infty, (H \circ X)_\infty \rangle] = E[\int^{\infty} H_r d[X, Y]_r]$. We conclude by showing that, for bounded H , $H \circ X = \int H dX$.

Theorem 2.2. *If G is a real, separable Hilbert space and X is a G -valued, square integrable martingale, then for all real-valued, bounded, optional processes $H, H \circ X = \int HdX$.*

Proof: By Theorem 1.2, it suffices to show $(H \circ X)_\infty = \int HdJ_X$ a.s. To do so, we apply a monotone class argument. We let M be the class of processes for which the result is true. Then M is a linear space and, by the proof of the preceding theorem, M , contains the class of processes $C = \{1_{[S, T]} : [S, T] \in \mathcal{S}\}$. Then, M also contains the constants.

If $\{H^n\}$ is a uniformly bounded sequence of non-negative processes in M which increase to H , or if $\{H^n\}$ is a sequence in M which converges uniformly to H , then $\int H^n dJ_X \xrightarrow{n} \int HdJ_X$ in $L_1(G)$. Also, $E[\int^\infty (H_r^n - H_r)^2 : d[X, X]_r] \xrightarrow{n} 0$. It follows that $(H^n \circ X)_\infty \xrightarrow{n} (H \circ X)_\infty$ in $L_2(G)$. Since $(H^n \circ X)_\infty = \int H^n dJ_X$ a.s. for each n , it follows that $(H \circ X)_\infty = \int HdJ_X$ a.s. By the monotone class theorem, M contains all bounded processes measurable with respect to $\sigma(C) = \mathcal{O}$. \square

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