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On Fibrations of Cylinderlike Surfaces

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Introduction

In this paper we consider fibrations of an affine, rational, nonsingular surface, $T = \operatorname{Spec} A$, over an algebraically closed field k of characteristic zero having the properties that $A^* = k^*$, $\operatorname{Pic} T$ is torsion, and T contains a nonempty open subset U which is isomorphic to $\mathbf{A}^1 \times C$ where C is a rational curve. We will call such a surface a cylinderlike surface and call the subset U a cylindrical open subset. Interest in these surfaces arose from a result of Miyanishi [M, Th. 0] which says that if T is a cylinderlike surface and $\operatorname{Pic} T = 0$, then $T \approx \mathbf{A}^2$. In certain applications of this result it is clear that $\operatorname{Pic} T$ is torsion and the main difficulty is to show that $\operatorname{Pic} T = 0$. This led to the consideration of cylinderlike surfaces. Although in general Miyanishi's theorem fails for these surfaces, several results concerning them have been obtained. One of these results, which is due to Richard Swan and which will be proved as Proposition 1.1 of this paper, says that if $T = \operatorname{Spec} A$ is a cylinderlike surface, then there are $\alpha_i \in k$, $n_i \in \mathbf{Z}^+$, and prime ideals P_i such that $k[s,t] \subseteq A \subseteq k\left[s, t, \frac{1}{\prod_{i=1}^r (s-\alpha_i)}\right]$ and $(s - \alpha_i) = P_i^{n_i}$. Thus, $T - \cup_{i=1}^r V((s-\alpha_i)) \approx \operatorname{Spec} k\left[s, t, \frac{1}{\prod_{i=1}^r (s-\alpha_i)}\right]$ is a cylindrical open set. From this it follows that $T \to \operatorname{Spec} k[s]$ gives a fibration of T having the properties that for each $\alpha \in k$ the fiber defined by $s - \alpha$ is irreducible and that the complement of a finite set of these fibers is a cylindrical open set. We call a fibration of T cylinderical if it arises from $k[x] \subset A$ and if the fibers defined by $x - \alpha$ have the two properties above. For $T = \operatorname{Spec} k[s,t]$ the cylindrical fibration given by Swan's result is not unique. We are interested in the question of whether a cylindrical fibration is unique for a cylinderlike surface T with $\operatorname{Pic} T \neq 0$. In this paper we show that this question has no simple answer by giving three examples of cylinderlike surfaces – two with multiple cylindrical fibrations and one with a unique cylindrical fibration.

We wish to thank Richard Swan and David Wright for making their unpublished work [S] and [W] available to us.

1. Preliminary Results

This first section gives the proofs of several unpublished results of Swan and Wright which will be used later in the paper.

Throughout the paper k will denote an algebraically closed field of characteristic zero, $v_D(f)$ will denote the valuation of a function f at a prime divisor D, and $l(M)$ will denote the length of a k-module M.

Proposition 1.1. *Let $T = \mathrm{Spec}\, A$ be a cylinderlike surface and let $U \approx \mathbf{A}^1 \times C$ be an open cylindrical subset. Then there are $\alpha_i \in k$, $n_i \in \mathbf{Z}^+$, and height one prime ideals P_i with*

$$k[s,t] \subset A \subset k\left[s, t, \frac{1}{\prod_{k=1}^{r}(s-\alpha_i)}\right], \quad U \approx \mathrm{Spec}\, k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$$

and $(s-\alpha_i) = P_i^{n_i}$.

Proof ([S]). By [I, Th. 2.15, p. 124], $T - U$ is pure of codimension 1. Thus, there are height one prime ideals P_i such that $T - U = \cup_{i=1}^{r} V(P_i)$. Since $\mathrm{Pic}\, T$ is torsion, there are $a_i \in A$ such that $(a_i) = P_i^{n_i}$. Thus, for $f = \prod_{i=1}^{r} a_i$ we have $T - U = V((f))$ and $U = \mathrm{Spec}\, A_f$.

Since C is rational, we have $C \subset \mathbf{P}^1$. By making C smaller if necessary we may assume that $C \subset \mathbf{A}^1$. Then the argument above gives $C = \mathrm{Spec}\, B$ where $B = k\left[\frac{1}{g(s)}\right]$. Thus, $U = \mathrm{Spec}\, B[t]$.

We now have $A_f = B[t]$. Let $R = A \cap B$. Since $f \in (A_f)^* = B^*$, we have $f \in R$ and $R_f = A_f \cap B_f = A_f \cap B = B$. Replace t by $f^n t$ so that $t \in A$. Since A and B are normal, R is also normal. Thus, R is an intersection of valuation rings of $k(s)$. These are $k[s^{-1}]_{(s^{-1})}$ and $k[s]_{(p)}$ where p is irreducible. Since the intersection of all of these valuation rings is k, at least one must be omitted from the intersection which forms R. Since k is algebraically closed, irreducible elements of $k[s]$ have the form $s - \alpha$ for $\alpha \in k$. Replacing s by $\frac{1}{s-\alpha}$ if necessary, we may assume that $k[s^{-1}]_{(s^{-1})}$ is omitted. If $k[s]_{(p)}$ is also omitted, then $p \in R^* \subset A^* = k^*$. Thus, $R = \bigcap_{p \text{ irreducible}} k[s]_{(p)} = k[s]$. Since $f \in R$, we may take $f = \prod_{i=1}^{r}(s - \alpha_i)$ with $(s-\alpha_i) = P_i^{n_i}$. Thus, $k[s,t] = R[t] \subset A \subset A_f = R_f[t] = k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$ and $U \approx \mathrm{Spec}\, k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$.

Proposition 1.2 ([W]). *In Proposition 1.1, we have $V((s-\alpha_i)) \approx \mathbf{A}^1$.*

Proof. Since $A^* = k^*$, we have that $k[s] \subset A$ induces a surjective morphism $\pi: T \to \mathbf{A}^1$ which, by a lemma of Miyanishi [M, Lemma 1], together with the fact that $\mathrm{Pic}\, T$ is torsion, has the property that each fiber, when reduced, is isomorphic to $\mathbf{A}^1$. Also, $k[s,t] \subset A$ gives a birational morphism $\rho\colon T \to V = \operatorname{Spec} k[s,t]$. For $i = 1, 2, \ldots, r$ let L_i be the fiber in V defined by $s - \alpha_i$. Since $\pi\colon T \to \mathbf{A}^1$ factors through V, we have that $\rho^{-1}(L_i)$ is a fiber of π. Thus, $\rho^{-1}(L_i) = V((s-\alpha_i)) \approx \mathbf{A}^1$.

Proposition 1.3. *Suppose* $k[s,t] \subset A \subset k\left[s, t, \frac{1}{\prod_{i=1}^{r}(s-\alpha_i)}\right]$ *with* $(s - \alpha_i) = P_i^{n_i}$ *for some* $n_i \in \mathbf{Z}^+$. *Then there is* $z \in A$ *such that* $k[s,z] \subset A \subset k\left[s, z, \frac{1}{\prod_{n_i \geq 2}(s-\alpha_i)}\right]$.

Proof (adapted from [S]). Suppose $n_1 = 1$. Let $a_1, a_2, \ldots, a_N$ generate A. There are nonnegative integers m_j such that $(s-\alpha_1)^{m_j} a_j \in k\left[s, t, \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right]$. We use induction on $\sum_{j=1}^{N} m_j$ to show that there is $z \in A$ such that $a_j \in k\left[s, z, \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right]$ for $j = 1, 2, \ldots, N$. Let D be the divisor corresponding to P_1. If $m_1 > 0$, then $(s-\alpha_1)^{m_1} a_1 = \frac{g(s,t)}{\prod_{i=2}^{r}(s-\alpha_i)}$ with $v_D(g) > 0$. Thus, $g(s,t) = (s-\alpha_1)g_1(s,t) + (t-\beta)g_0(t)$ with $v_D(t-\beta) > 0$. Since $v_D(s-\alpha_1) = n_1 = 1$, we have that $s - \alpha_1$ divides $t - \beta$ in A. Let $z' = \frac{t-\beta}{s-\alpha_1}$. Then $k[s,z'] \subset A$ and $(s-\alpha_1)^{m_1-1} a_1 \in k\left[s, z', \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right]$. Thus, $\sum_{j=1}^{N} m_j$ is reduced and there is $z \in A$ with $k[s,z] \subset A \subset k\left[s, z, \frac{1}{\prod_{i=2}^{r}(s-\alpha_i)}\right]$.

Proposition 1.4. *In Proposition 1.3,* $\operatorname{Pic} T \approx \coprod_{i=1}^{r} \mathbf{Z}/n_i\mathbf{Z}$ *for* $T = \operatorname{Spec} A$.

Proof. Let $U = T - \cup_{i=1}^{r} V((s-\alpha_i))$ and let D_i be the prime divisor corresponding to P_i. By [H, Prop. 6.5, p. 133], together with the fact that $\operatorname{Pic} U = 0$, we have that $\operatorname{Pic} T$ is generated by $D_1, D_2, \ldots, D_r$. Let $\rho\colon T \to V = \operatorname{Spec} k[s,t]$ be the morphism given by $k[s,t] \subset A$. Then ρ is an isomorphism from U to $V - \cup_{i=1}^{r} V((s-\alpha_i))$. Thus if $f \in k(s,t)^*$, and if the support of the divisor of f is contained in $\cup_{i=1}^{r} V((s-\alpha_i))$ in T, then the support of the divisor of f is also contained in $\cup_{i=1}^{r} V((s-\alpha_i))$ in V. Therefore, $f = \prod_{i=1}^{r}(s-\alpha_i)^{e_i}$ and the divisor of f is an element of $\langle n_1D_1, n_2D_2, \ldots, n_rD_r\rangle$. Thus, $\operatorname{Pic} T = \frac{\langle D_1, D_2, \ldots, D_r\rangle}{\langle n_1D_1, n_2D_2, \ldots, n_rD_r\rangle} = \coprod_{i=1}^{r} \mathbf{Z}/n_i\mathbf{Z}$.

2. Examples

Example 2.1. Let $T = \operatorname{Spec} A$ where $A = k\left[s, t, \frac{t^2}{s}, \frac{t(t^2-s)}{s^2}, \frac{(t^2-s)^2}{s^3}\right]$. Then T is a cylinderlike surface with multiple cylindrical fibrations.

Proof. Clearly T is an affine, rational surface. If $U = T - V((s))$, then $U = \operatorname{Spec} k\left[s, t, \frac{1}{s}\right]$ so that U is a cylindrical open subset of T.

Let P be a height one prime ideal containing s. For convenience, let $u = \frac{(t^2-s)^2}{s^3}$, $v = \frac{t^2}{s}$, and $w = \frac{t(t^2-s)}{s^2}$. Then $sv = t^2$, $su = (v-1)^2$, and $(v-1)u = w^2 - u$ so that t, $v-1$, and w^2-u are elements of P. Since $\frac{A}{(s,t,v-1,w^2-u)A} = k[\bar{w}]$, we have $P = (s,t,v-1,w^2-u)A$. In A_P we have

$$(1) \qquad s = \frac{t^2}{v}, \quad v - 1 = \frac{tw}{v}, \quad \text{and} \quad w^2 - u = \frac{tuw}{v}.$$

Thus, $PA_P = (t)A_P$. Also, if D is the prime divisor corresponding to P, then the divisor of s is $2D$ so that by Proposition 1.4, Pic $T\mathbf{Z}/2\mathbf{Z}$.

To show that $A^* = k^*$, we use an argument due to Swan [S]. We embed A into $k[x,y]$ by $s \to x^2$, $t \to x(1+yx^2)$, $v \to x^2y(2+x^2y)+1$, $u \to x^2y^2(2+x^2y)^2$ and $w \to xy(1+x^2y)(2+x^2y)$. This is an injection because the image is 2-dimensional.

Since $U \approx \operatorname{Spec} k\left[s,t,\frac{1}{s}\right]$ we have that T is nonsingular at any point of U. Let $p \in V((s))$ and let m be the maximal ideal corresponding to p. Then $m = (s,t,v-1,w^2-u,w-\beta)A$ for some $\beta \in k$, and from (1), $mA_m = (t,w-\beta)A_m$. Thus, if $\tilde{m} = mA_m$, we have $\dim \tilde{m}/\tilde{m}^2 \leq 2 = \dim A$. Therefore, T is nonsingular at the point corresponding to m.

Finally, we note that $\frac{(w^2-u)^2}{u^3} = s$, $\frac{w^2}{u} = \frac{t^2}{s}$, and $\frac{w(w^2-u)}{u^2} = t$. Hence, $A = k\left[u, w, \frac{w^2}{u}, \frac{w(w^2-u)}{u^2}, \frac{(w^2-u)^2}{u^3}\right]$ and $k[u] \subset A$ gives a second cylindrical fibration of T.

The two cylindrical fibrations of Example 2.1 differ by the automorphism of A given by $s \to u$ and $t \to w$. The next example shows that not all distinct cylindrical fibrations of a cylinderlike surface differ by an automorphism.

Example 2.2. Let $T = \operatorname{Spec} A$ where $A = k\left[s, t, \frac{t^3}{s}, \frac{t^2(t^3-s)}{s^2}, \frac{(t^3-s)^3}{s^4}\right]$. Then T is a cylinderlike surface with two cylindrical fibrations which do not differ by an automorphism.

Proof. Let $v = \frac{t^3}{s}$, $w = \frac{t^2(t^3-s)}{s^2}$, and $x = \frac{(t^3-s)^3}{s^4}$. An argument similar to the one used in Example 2.1 shows that T is a cylinderlike surface with Pic $T \approx \mathbf{Z}/3\mathbf{Z}$, that the only height one prime ideal containing s is $P = (s,t,v-1,w^3-x)$ with $A/P = k[\bar{w}]$, and that if D is the prime divisor corresponding to P, then $v_D(t) = 1$, $v_D(s) = 3$, and, since $(t^3-s)^3 = s^4w$, $v_D(t^3-s) = 4$.

Let $y = \frac{t(t^3-s)^2}{s^3}$. Then $y \in A$ since the divisor of y is effective. Also, $A = k\left[x, y, \frac{y^2}{x}, \frac{y^3}{x^2}, \frac{y(y^3-x^2)}{x^3}, \frac{(y^3-x^2)^3}{x^7}\right]$ so that $k[x] \subset A$ gives a second cylindrical fibration of T.

Now suppose that the fibrations given by $k[s] \subset A$ and $k[x] \subset A$ differ by an automorphism φ of A. Then $V((s)) = V((\varphi(x-\lambda)))$ for some $\lambda \in k$. As in Proposition 1.4, the divisor of $\varphi(x-\lambda)$ is $3eD$. Since $A^* = k^*$, this gives us $\beta \in k^*$ with $\varphi(x-\lambda) = \beta s^e$. Since $V((\varphi(x-\gamma)))$ is irreducible for all

$\gamma \in k$, we must have $e = 1$. Thus, we may assume $\varphi(s) = x$. Since $k[s,t] \subset A$ gives a morphism $\rho: T \to V = \operatorname{Spec} k[s,t]$ which is an isomorphism on $T - V((s))$, we have that for $f \in k[s,t]$, the divisor of f in $\operatorname{Div} T$ differs from the divisor of f in $\operatorname{Div} V$ only at the prime divisor D. Thus, there are prime divisors C and E with $(t) = C + D$ and $(t^3 - s) = E + 4D$. Therefore, $(x) = 3E$ and there must be other prime divisors C' and E' with $(\varphi(t)) = C' + E$ and $(\varphi(t^3 - s)) = E' + 4E$. We have that $C' \neq D$, since otherwise, $\varphi(t) = \frac{\gamma(t^3-s)}{s}$ for $\gamma \in k^*$, and $v_E(\varphi(t^3 - s)) = v_E\left(\frac{(\gamma^3 s-1)(t^3-s)^3}{s^4}\right) \neq 4$. Thus, $\varphi(t) = \frac{g(t^3-s)}{s^n}$ and $\varphi(t^3 - s) = \frac{h(t^3-s)^4}{s^{3n}}$ for some irreducible g and h in $k[s,t]$ with $g^3 - s^{3n-4} = h(t^3 - s)$ and with $v_D(g) = 3n - 4 \geq 2$ and $v_D(h) = 9n - 16 \geq 2$.

Similarly, the preimage of t has the form $\frac{f}{s^r}$ for some $f \in k[s,t]$ with $v_D(f) = 3r$. Write $f = \sum_{i=1}^{R} \gamma_i s^{a_i} t^{b_i}$ with $\gamma_i \in k^*$. Then

$$t = \varphi\left(\frac{f}{s^r}\right) = \left(\sum_{i=1}^{R} \gamma_i \frac{g^{b_i}(t^3 - s)^{3a_i + b_i}}{s^{4a_i + nb_i}}\right)\left(\frac{s^{4r}}{(t^3 - s)^{3r}}\right).$$

Let $M = \max_i\{4a_i + nb_i\} \geq 4r$ and let $I = \{i \mid 4a_i + nb_i = M\}$. If $M > 4r$, then s divides $\sum_{i \in I} \gamma_i g^{b_i}(t^3 - s)^{3a_i + b_i}$. Setting $p = \frac{n}{gcd(n,4)}$ and $q = \frac{4}{gcd(n,4)}$, we have that there is $\beta \in k^*$ such that s divides $g^q - \beta(t^3 - s)^{3p-q}$ and $g^q - \beta(t^3 - s)^{3p-q}$ divides $\sum_{i \in I} \gamma_i g^{b_i}(t^3 - s)^{3a_i + b_i}$. Thus, for $g = sg_1(s,t) + g_0(t)$, we have that $(g_0(t))^q = \beta t^{9p-3q}$. Therefore, q divides $9p$ so that $q = 1$, $n = 4p$, and $g = sg_2(s,t) + \beta(t^3 - s)^{3p-1}$. Let ψ be the automorphism of A given by $s \to s$ and $t \to t - \beta s^p$ and let $\varphi' = \varphi \circ \psi$. Then $\varphi'(s) = x$, $\varphi'(t) = \frac{g_2(t^3-s)}{s^{n-1}}$, and $t = \varphi'\left(\frac{f'(s,t)}{s^r}\right)$ where $f'(s,t) = \sum_{i=1}^{R} \gamma_i s^{a_i}(t + \beta s^p)^{b_i} = \sum_{j=1}^{Q} \lambda_j s^{a_j'} t^{b_j'}$. Let $M' = \max_j\{4a_j' + (n-1)b_j'\}$. For each $j \in \{1, 2, \ldots, Q\}$ there is an $i \in \{1, 2, \ldots, R\}$ such that $a_j' = a_i + p(b_i - b_j')$. Thus, $4a_j' + (n-1)b_j' = 4a_i + nb_i - b_j'$. Since $g - \beta(t^3 - s)^{3p-1}$ divides $\sum_{i \in I} \gamma_i g^{b_i}(t^3 - s)^{3a_i + b_i}$, we have that $t - \beta s^p$ divides $\sum_{i \in I} \gamma_i s^{a_i} t^{b_i}$ and t divides $\sum_{i \in I} \gamma_i s^{a_i}(t + \beta s^p)^{b_i}$ so that if $i \in I$, then $b_j' > 0$. Therefore, $M' < M$ and we may assume that $M = 4r$.

Now write $f = \sum_{j=1}^{L} \xi_j s^{c_j} t^{d_j}(t^3 - s)^{e_j}$ with $\xi_j \in k^*$ and $0 \leq d_j < 3$. Let $m = \min_j\{3c_j + d_j + 4e_j\}$ and let $J = \{j \mid 3c_j + d_j + 4e_j = m\}$. If $i, j \in J$ with $e_i = e_j$, then 3 divides $d_i - d_j$ so that $d_i = d_j$ and $c_i = c_j$. Thus, we may assume that $e_i \neq e_j$ for $i \neq j$. Choose $l \in \{0, 1, 2\}$ with $m + l = 3k$ for some $k \in \mathbf{Z}^+$. If $j \notin J$, then $v_D\left(s^{c_j} t^{l+d_j}(t^3 - s)^{e_j}\right) > 3k$ so that $\frac{s^{c_j} t^{l+d_j}(t^3-s)^{e_j}}{s^k} \in P$. Thus,

$$\frac{t^l f}{s^k} = a + \sum_{j \in J} \xi_j \frac{t^{l+d_j}(t^3 - s)^{e_j}}{s^{k-c_j}} = a + \sum_{j \in J} \xi_j v^{k-c_j-2e_j} w^{e_j} \tag{2}$$

with $a \in P$. In A/P, $\bar{v} = 1$ so $\overline{\frac{t^l f}{s^k}} = \sum_{j \in J} \xi_j \bar{w}^{e_j} \neq 0$. Therefore, $m = v_D(f) = 3r$. Let $N = \max_j\{4c_j + nd_j + 3ne_j\}$. An argument similar to the one used above shows that $N = M = 4r$. Thus,

$$t = \varphi\left(\frac{f(s,t)}{s^r}\right) = \left(\sum_{j=1}^{L} \xi_j \frac{g^{d_j} h^{e_j} (t^3 - s)^{3c_j + d_j + 4e_j}}{s^{4c_j + nd_j + 3ne_j}}\right)\left(\frac{s^{4r}}{(t^3 - s)^{3r}}\right)$$
$$= \sum_{j=1}^{L} \xi_j s^{M - 4c_j - nd_j - 3ne_j} g^{d_j} h^{e_j} (t^3 - s)^{3c_j + d_j + 4e_j - 3r}.$$

But this is impossible since $v_D(s)$, $v_D(g)$, $v_D(h)$, and $v_D(t^3 - s)$ are all greater than $v_D(t)$.

The third example, which is of a cylinderlike surface with a unique cylindrical fibration, was used by Swan [S] to show that Miyanishi's result does not hold if the condition that $\operatorname{Pic} T = 0$ is replaced by the weaker condition that $\operatorname{Pic} T$ is torsion.

Example 2.3. Let $T = \operatorname{Spec} A$ where $A = k\left[s, t, \frac{t^2 - s}{s^2}\right]$. Then T is a cylinderlike surface whose cylindrical fibration is unique.

Proof. As in Example 2.1, T is a cylinderlike surface with $\operatorname{Pic} T \approx \mathbf{Z}/2\mathbf{Z}$. The only height one prime ideal containing s is $(s,t)A$ and $A/(s,t)A = k[\bar{u}]$ where $u = \frac{t^2 - s}{s^2}$. If D is the divisor corresponding to $(s,t)A$, then, since $s = \frac{t^2}{su + 1}$, we have $(s,t)A_{(s,t)} = (t)A_{(s,t)}$ so that $v_D(t) = 1$, $v_D(s) = 2$, and $v_D(t^2 - s) = v_D(s^2 u) = 4$.

Suppose there is a second cylindrical fibration of T given by $k[x] \subset A$. Propositions 1.1, 1.3, and 1.4, together with the fact that $\operatorname{Pic} T \approx \mathbf{Z}/2\mathbf{Z}$, give us elements z and w of A such that $k[z, w] \subset A \subset k\left[z, w, \frac{1}{z}\right]$, the divisor of z is $2C$ for some prime divisor C, and $V((z))$ is the fiber of $T \to \operatorname{Spec} k[x]$ defined by $x - \gamma$ for some $\gamma \in k$. An argument similar to the one used in Example 2.2 to show that we could take $\varphi(s) = x$ gives us that if $C = D$, then $x - \gamma\beta = s$ for some $\beta \in k^*$. Thus, the second fibration which is given by $k[x] \subset A$ is the same as the original fibration given by $k[s] \subset A$. Therefore, $C \neq D$.

Let Q be the height one prime ideal corresponding to C. By Proposition 1.2, we have $A/Q \approx k[y]$. We will show that there is $\alpha \in k^*$ such that $\deg(\bar{s}) \neq \deg(\bar{s} - \alpha)$. By [H, Prop. 6.5, p. 133], we have an exact sequence $\mathbf{Z} \to \operatorname{Pic} T \to \operatorname{Pic} \operatorname{Spec} k\left[z, w, \frac{1}{z}\right] = 0$ where the first homomorphism is given by $1 \to 1 \cdot C$. Thus, C is not principal, but $2C$ is principal. Let f be an irreducible element of $k[s,t]$ with $Q \cap k[s,t] = (f)k[s,t]$. Then the divisor of f is $C + v_D(f)D$. We must have that $v_D(f)$ is odd, since otherwise $v_D(f) = 2n$ and the divisor of $\frac{f}{s^n}$ is C. Say $v_D(f) = 2n + 1$.

As in Example 2.2, we can write $f = \sum_{j=1}^{N} \xi_j s^{a_j} t^{b_j} (t^2 - s)^{c_j}$ with $\xi_j \in k^*$, with $0 \leq b_j \leq 1$, and with $\min_j\{2a_j + b_j + 4c_j\} = v_D(f) = 2n + 1$. If $a_j = 0$, then $b_j + 2c_j > n$. Thus, $\deg f(0,t) > n$ and there is $\alpha \in k^*$ with $\deg f(\alpha, t) > n$. Since $\rho\colon T \to \operatorname{Spec} k[s,t]$ is an isomorphism on $T - V((s))$, we have that $A/(s-\alpha)A \approx \frac{k[s,t]}{(s-\alpha)k[s,t]}$. Thus, in A/Q we have

$$\deg(\bar{s}-\alpha) = l\left(\frac{A/Q}{(\bar{s}-\alpha)A/Q}\right) = l\left(\frac{k[s,t]}{(s-\alpha,f)}\right) = \deg\ f(\alpha,t) > n.$$

Now we wish to find $\deg(\bar{s})$ in A/Q. Let $\hat{m}$ be a maximal ideal of A/Q which contains $\bar{s}$ and let π be the usual homomorphism from A to A/Q. Let $\pi^{-1}(\hat{m}) = m$, a maximal ideal of A containing s. Since $A/(s,t)A = k[\bar{u}]$, we have $m = (s,t,u-\beta)$ for some $\beta \in k$. Since $Q \cap k[s,t] = (f)k[s,t]$, since the divisor of s is $2D$, and since $v_D(f) = 2n+1$, we have

$$Q = \left\{ \frac{gf}{s^p} \,\middle|\, v_D(g) \geq 2p-2n-1 \right\}.$$

In particular, $\frac{tf}{s^{n+1}} \in Q \subset m$. As in (2) of Example 2.2, if $J = \{j | 2a_j + b_j + 4c_j = 2n+1\}$, then $\frac{tf}{s^{n+1}} = a' + \sum_{j\in J} \gamma_j (su+1)u^{c_j} = a + \varphi(u)$ for some $a, a' \in (s,t)A$. Thus, $u-\beta$ divides $\varphi(u)$. Say $\varphi(u) = (u-\beta)^e \varphi_1(u)$. Then $s = \frac{t^2}{su+1}$ and $(u-\beta)^e = \frac{1}{\varphi_1(u)}\left(\frac{tf}{s^{n+1}} - a\right)$, so $(s,t,(u-\beta)^e)A_m \subset (t,Q)A_m$. On the other hand, suppose $\frac{gf}{s^p} \in Q$. If $v_D(g) > 2p-2n-1$, then $\frac{gf}{s^p} \in (s,t)A$. If $v_D(g) = 2p-2n-1$, then $\frac{tg}{s^{p-n}} = b + \psi(u)$ with $b \in (s,t)A$. Thus, $(su+1)\frac{gf}{s^p} = \left(\frac{tf}{s^{n+1}}\right)\left(\frac{tg}{s^{p-n}}\right) = c + (u-\beta)^e \varphi_1(u)\psi(u)$ with $c \in (s,t)A$. Therefore, $\frac{gf}{s^p} \in (s,t,(u-\beta)^e)A_m$.

Now let $Y = \operatorname{Spec} A/Q$ and let y be the point of Y corresponding to $\hat{m}$. Then

$$\begin{aligned} v_{y,Y}(\bar{s}) = 2v_{y,Y}(\bar{t}) &= 2l\left(\frac{(A/Q)_{\hat{m}}}{(t)(A/Q)_{\hat{m}}}\right) = 2l\left(\frac{A_m}{(t,Q)A_m}\right) \\ &= 2l\left(\frac{A_m}{(s,t,(u-\beta)^e)A_m}\right) = 2l\left(\frac{k[s,t,u]_{(s,t,u-\beta)}}{(s,t,(u-\beta)^e)}\right) = 2e. \end{aligned}$$

If we let $\varphi(u) = \prod_{i=1}^k (u-\beta_i)^{e_i}$, then $\deg(\bar{s}) = 2\sum_{i=1}^k e_i = 2\ \deg \varphi(u)$. But if $j \in J$, then $b_j = 1$ and $2c_j \leq n$ so that $2\ \deg \varphi(u) \leq n$. Since we cannot have $\deg(\bar{s}) \leq n$ and $\deg(\bar{s}-\alpha) > n$, there is no second cylindrical fibration of T.

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