Ulam Quarterly — Volume 1, Number 1, 1992

Alexandre Grothendieck's EGA V
Part I: Hyperplane Sections and Conic Projections(1)
(Interpretation and Rendition of his 'prenotes')

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§1 Preliminaries and Notation

Let S be a prescheme, $\mathcal{E}$ be a locally free Module of finite type over S, and $\check{\mathcal{E}}$ be its dual. We denote by $P = \mathcal{P}(\mathcal{E})$ the projective fibration defined by $\mathcal{E}$ and by $\check{P}$ the projective fibration defined by $\check{\mathcal{E}}$. $\check{P}$ will be called the *scheme of hyperplanes* of P. This terminology can be justified as follows. Let ξ be a section of $\check{P}$ over S which is therefore determined by an invertible quotient module $\mathcal{L}$ of $\check{\mathcal{E}}$. From it we obtain an invertible quotient module $\mathcal{L}_P$ of $\check{\mathcal{E}}_P = (\mathcal{E}_P)\check{}$, on the other hand, we have the invertible quotient module $\mathcal{O}_p(1)$ of $\mathcal{E}_p$. Passing to the duals we may take $\mathcal{L}_P^{-1}$ (resp. $\mathcal{O}_P(-1)$) to be invertible submodules (locally *direct factors*) of $\mathcal{E}_P$ (resp. of $(\mathcal{E}_P)\check{}$) and the pairing $\mathcal{E}_P \otimes \check{\mathcal{E}}_P \to \mathcal{O}_P$ defines therefore a natural pairing

$$\mathcal{O}_P(-1) \otimes \mathcal{L}_P^{-1} \longrightarrow \mathcal{O}_P \qquad (*)$$

or also a transposed homomorphism

$$\mathcal{O}_P \longrightarrow \mathcal{O}_P(1) \otimes \mathcal{L}_P = \mathcal{L}_P(1) \qquad (**)$$

i.e. a section of $\mathcal{L}_P(1)$ canonically defined by ξ. The "divisor" of that section, i.e. the closed subscheme H_ξ of P defined by the Ideal, image of $(*)$, is called the *hyperplane in* P defined by the element $\xi \in \check{P}(S)$. We could also describe it by noting that locally over S, ξ is given by a section φ of $\mathcal{E}$ such that $\varphi(s) \neq 0$ for all s (φ is determined by ξ up to multiplication by an invertible section of $\mathcal{O}_S$); since $\mathcal{E} = p_*(\mathcal{O}_p(1))$, ($p$: $P \to S$ is the projection), φ can be considered as a section of $\mathcal{O}_P(1)$, the divisor of which is nothing else but H_ξ.

Of course, if we consider $\mathcal{L}^{-1}$ as an invertible submodule of $\mathcal{E}$ locally a direct factor in $\mathcal{E}$ then the correspondence between ξ (i.e. $\mathcal{L}$ or $\mathcal{L}^{-1} \subset \mathcal{E}$) and φ is obtained by taking for φ a section of $\mathcal{L}^{-1}$ which does not vanish at any point, i.e. by a trivialization of $\mathcal{L}^{-1}$ (which exists locally anyway). Let us note that H_ξ is simply $P(\mathcal{E}/\mathcal{L}^{-1})$ (canonical isomorphism), which is a third way to describe H_ξ (N.B. $P(\mathcal{E}/\mathcal{L}^{-1})$ is indeed canonically embedded in $P = \mathcal{P}(\mathcal{E})$ which has the advantage of proving in addition that H is a projective fibration over S and is à fortiori smooth over S. (Again it would have been better to say in sect. 17 of EGA IV that a projective fibration is smooth...). It would be best to begin this way.

Remarks. The construction of H_ξ in terms of ξ is compatible with base change, as one can see right away, in other words one finds a homomorphism of functors $(\mathcal{S}ch)^o/_S \to (\mathcal{E}ns)$, $\check{P} \to \mathcal{D}iv(P/S)$ where the second term denotes the functor of "relative divisors" of P/S whose values at S' (an arbitrary S prescheme) is the set of closed subschemes of $P_{S'}$ which are complete transversal intersections and of codimension 1 relatively to S' (cf. sect. 19) [of EGA IV, Intepreter].[1]

It is easy to show that this functor homomorphism is a monomorphism, in other words that ξ is determined by H_ξ. (This last fact justifies the terminology "scheme of hyperplanes" used above.) We shall see that the functor $\mathcal{D}iv(P/S)$ is representable by the prescheme (direct) sum of $P(Symm^k(\check{\mathcal{E}}))$ so that $\check{P}$ can be identified to an open and closed subscheme of $\mathcal{D}iv(P/S)\ldots$[2] (N.B. Let's remark that, the determination of the relative divisors of P/S could be done with the means available right now, using results of Ch. III and could be added as an example to sec. 19 ...) [of EGA IV, Interp.].

Let us now make the base change $S' = \check{P} \to S$ and let us consider the diagonal section (or "generic section") of $\check{P}_{S'} = \mathcal{P}(\check{\mathcal{E}}_{S'})$ over S': we find a closed subscheme $\mathcal{H}$ of $P_{S'} = P \times_S \check{P}$, called sometimes the *incidence scheme* between P and $\check{P}$ defined by the Ideal image of the canonical homomorphism

$$\mathcal{O}_P(-1) \otimes_S \mathcal{O}_{\check{P}}(-1) \longrightarrow \mathcal{O}_{P \times_S \check{P}};$$

from what we know already, it is a projective fibration over $\check{P}$, and by symmetry it is also a projective fibration over P. We recover, of course, the

[1] Uses notation of new edition of EGA IV [Interp.]

[2] Compare with Mumford's: 'Lectures on curves on an algebraic surface.' [Interp.]

"special" hyperplanes H_ξ (for ξ a section of $\check{P}$ over S) by starting out from the "universal hyperplane" H and by taking its inverse image for the base change $S \xrightarrow{\xi} \check{P}$.

The same remark holds for every point ξ of $\check{P}$ with values in any S-prescheme S' which (considered as a section of $P_{S'}$ over S') allows us to define an $H_\xi \subset P_{S'}$; the latter is nothing else but the inverse image of H by the base change $S' \xrightarrow{\xi} \check{P}$.

In what follows we assume a prescheme X of finite type over S and an S morphism $f\colon X \to P$. One of the main objectives of this section is to study for every hyperplane H_ξ of P, where $\xi \in P(S)$, its inverse image

$$Y_\xi = f^{-1}(H_\xi) = X \times_P H_\xi$$

and especially to relate the properties of X and Y_ξ. As usual we consider $P(S')$, S' an arbitrary S scheme (in this case H_ξ is a hyperplane in $P_{S'}$) and we put again

$$Y_\xi = f_{S'}^{-1}(H_\xi) = X_{S'} \times_{P_{S'}} H_\xi = X \times_P H_\xi,$$

where the subscript S' denotes as usual the effect of the base change $S' \to S$ and where in the last expression we consider H_ξ as a P-prescheme via the composite morphism $H_\xi \to P_{S'} \to P$. It is therefore again convenient to consider the case where ξ is "universal" i.e. where $S' = \check{P}$ and ξ is the diagonal section so that $H_\xi = H$, in this case one observes (up to better notations to be suggested by Dieudonné) that $Y = Y_\xi$. In the general case of a $\xi\colon S' \to \check{P}$, one has therefore also $Y_\xi = Y \times_{\check{P}} S'$. Finally if $\mathcal{F}$ is a *sheaf of modules*[3] over X we denote by G_ξ its inverse image over Y_ξ by $\mathcal{G}$ its inverse image over $\mathcal{H}$ so that we also have $G_\xi = \mathcal{G} \otimes_{\mathcal{O}_{\check{P}}} \mathcal{O}_{S'}$.

Let us summarize in a small diagram the essentials of the constructions and notations considered.

$$\begin{array}{ccccccc}
\mathcal{F} & & & & \mathcal{G} & & G_\xi \\
X & \longleftarrow & X \times_S \check{P} & \longleftarrow & Y & \longleftarrow & Y_\xi \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \longleftarrow & P \times_S \check{P} & \longleftarrow & \mathcal{H} & \longleftarrow & H_\xi \\
\downarrow & & \downarrow & \swarrow & & \swarrow & \\
S & \longleftarrow & \check{P} & \longleftarrow & S' & &
\end{array}$$

(The squares and diamonds appearing in this diagram are Cartesian).

In the next section we will study systematically the following case: S' is the spectrum of a field K and its image in $\check{P}$ is generic in the corresponding

[3] Ask A.G. If module always means coherent or quasi-coherent sheaf of modules.

fiber $\check{P}_s$. After making the base change $Spec\ k(s) \to S$ we are reduced to the case where S is the spectrum of a field k, what we are going to assume in the next section. Also most of the properties studied for X and Y_ξ are of "geometric nature" and therefore invariant under base field change, this allows us also (without loss of generality) to restrict ourselves to the case where K is algebraically closed or to the case where $K = k(\eta)$, η being the generic point of $\check{P}$ and $\xi\colon Spec(K) \to \check{P}$ is obviously the canonical morphism. We also note that for geometric questions concerning X, Y_ξ we can (after making a base change on k) restrict ourselves to the case of k algebraically closed.

A terminological note: If f is an immersion we usually call Y_ξ a *hyperplane section* of X (relatively to the projective immersion f and to the hyperplane H_ξ [Interp.]). There is no reason why not to extend this terminology to the case of an arbitrary f.

§2 Study of the generic hyperplane section: local properties

Let us recall that now $S = Spec(k)$, where k is a field. If η is a point of $\check{P}$ and if $\xi\colon Spec\ k(\eta) \to \check{P}$ is the canonical morphism we are also going to write H_η, Y_η, G_η instead of H_ξ, Y_ξ, G_ξ.

In this section η will always denote the generic point of $\check{P}$.

Proposition 2.1. *Let us assume that X is irreducible. Then Y_η is irreducible or empty and in the first case it dominates X; anyway $\overline{Y}$ is irreducible.*

Indeed, since $\mathcal{H} \to P$ is a *projective fibration* as it is for $Y \to X$ which implies that Y is irreducible if X is irreducible. So the generic fiber Y_η [Interp.] of Y over $\check{P}$ is irreducible or empty and in the first case its generic point is the generic point of Y which therefore lies over the generic point of X. q.e.d.

Proposition 2.2. *Let Z be a subset of P. Then its inverse image Z_η in H_η is empty if and only if every point of Z is closed. In particular if Z is constructible then $Z_\eta = \phi$ if and only if Z is finite.*

We may suppose that Z is reduced to a single point z and we only have to prove that the image of H_η in P consists exactly of the non-closed points of P. Let X be the closure of z, using 2.1 we only have to prove that $Z_\eta = \phi$ if and only if X is finite (X being a closed subscheme of P). Replacing X by $X_{k(\eta)} \hookrightarrow P_{k(\eta)}$ the 'only if' [French 'il faut' or necessary] part results from the following fact for which we have to have a reference and which fact deserves to be restated here as a lemma: if Y is *any* hyperplane section of X and if $Y_\eta = \phi$ then X is finite (indeed $X \subset P - H$ is affine and projective...). The 'sufficient' part is obvious, for example, by noticing that Y is a projective fibration of relative dimension $(n-1)$ over X (n being the relative dimension of P and $\check{P}$ over S), thus X being finite over k, Y

is of absolute dimension $n-1$, $\langle n = \dim P \rangle$ thus the morphism $Y \to \check{P}$ cannot be dominant thus its generic fiber Y_η is empty.

Corollary 2.3. *Let $f\colon X \to P$ be a morphism of finite type and let Z be a constructible subset of X. In order for its inverse image in Y_η to be empty it is necessary and sufficient for the image $f(Z)$ to be finite. In particular, in order for Y_η to be empty it is necessary and sufficient for $f(X)$ to be finite.*

Corollary 2.4. *Let Z, Z' be two closed subsets of X with Z irreducible, and let Z_η and Z'_η be their inverse images in Y_η. In order to have $Z_\eta \subset Z'_\eta$ it is necessary and sufficient for $f(Z)$ to be finite or to have $Z \subset Z'$. In order that $Z_\eta = Z'_\eta$ it is necessary and sufficient for $f(Z)$ and $f(Z')$ to be finite or to have $Z = Z'$.*

This is an immediate consequence of 2.3 as we see that $f(Z - Z \cap Z')$ can only be finite if $Z \subset Z'$ or if $f(Z)$ is finite (if we do not have $z \subset Z'$ then $z - Z \cap Z'$ is dense in Z, thus $f(Z - Z \cap Z')$ is dense in $f(Z)$, and if the former is finite and thus closed—being constructible—so is also the latter.

Corollary 2.5. *To every irreducible component X_i of X such that* $\dim \overline{f(X_i)} > 0$ *we assign its inverse image $Y_{i\eta}$ in Y_η. Then $Y_{i\eta}$ is an irreducible component of Y_η and we obtain this way a one-to-one correspondence between the set of irreducible components X_i of X such that* $\dim \overline{f(X_i)} > 0$ *and the set of irreducible components of Y_η.*

Indeed, it follows from 2.3 that Y_η is the union of all $Y_{i\eta}$ defined above, that are closed and non-empty subsets of Y; they are also irreducible because of 2.1. Finally, they are mutually not included in each other because of 2.4, hence the conclusion.

Let us notice that if $\dim X_i = d_i$ we have $\dim Y_i = d_i - 1$. More generally:

Proposition 2.6. *Let us assume that for every irreducible component X_i of X we have* $\dim \overline{f(X_i)} > 0$, *i.e. $Y_{i\eta} \neq \emptyset$, or that f is an immersion and* $\dim \overline{f(X)} > 0$. *Then we have* $\dim Y_\eta = \dim X - 1$.

We are reduced to the case where X is irreducible, since 2.5. By the very construction, Y_η is defined from $X_{k(\eta)}$ as the divisor of a section of an invertible module over $X_{k(\eta)}$ (i.e. the inverse image of $\mathcal{O}_P(1)$). On the other hand, $X_{k(\eta)}$ is irreducible (since X is irreducible and since $k(\eta)$ is a pure transcendental extension of k —fact that should have been mentioned at the beginning of the section ...) $\langle$ precision not given by A.G. $\rangle$ and $Y_\eta \neq X_{k(\eta)}$ since the image of Y_η in X (not like $X_{k(\eta)}$, which is faithfully flat over X) is not equal to X, indeed it does not contain the closed points of X because of 2.3. It follows that $\dim Y_\eta = \dim X_{k(\eta)} - 1 = \dim X - 1$ (reference needed for the last equality.) q.e.d.

Proposition 2.7. *Let $\mathcal{F}$ be a quasi coherent module over X, hence $\mathcal{G}_\eta$ over Y_η. Let Z_i be the associated prime cycles of $\mathcal{F}$ such that* $\dim \overline{f(Z_i)} > 0$. *Let $Z_{i\eta}$ be the inverse image of Z_i in Y_η, then the $Z_{i\eta}$ are exactly all the prime cycles associated with $\mathcal{G}_\eta$. Also, their inclusion relations are the same as those of Z_i.*

The last assertion is contained in 2.4. On the other hand, since $Y \to X$ is a projective fibration, thus flat with fibers (S_1) and irreducible, it follows from sect. 3 of EGA IV that the associated prime cycles with the inverse image $\mathcal{G}$ of $\mathcal{F}$ over Y are the inverse images of the associated prime cycles of $\mathcal{F}$. Hence they are induced on the generic fiber Y_η of Y over $\check{P}$, and the associated prime cycles to $\mathcal{G}_\eta$ are the *non-empty* inverse images of the Z_i which proves 2.7 by means of 2.3.

Actually we did not need Y but we could have used directly the fact that $Y_\eta \to X$ is flat with fibers (S_1) (and also geometrically regular, i.e. the morphism is regular) and with irreducible fibers (and even geometrically irreducible: they are localizations of projective schemes); same remark for the proof of 2.1.

Proposition 2.8. *Let $\mathcal{F}$ be coherent over X, let $y \in Y_\eta$, and x be its image in X. Let $P(M)$ be one of the following properties for a finitely generated module M over a noetherian local ring A:*

(i) coprof $M \leq n$ (ref)
(ii) M satisfies (S_k) (ref)
(iii) M is Cohen-Macaulay
(iv) M is reduced (ref)
(v) M is integral (ref)

Then for $\mathcal{G}_{\eta,y}$ to satisfy the property P, it is necessary and sufficient that $\mathcal{F}_x$ also satisfies it.

This follows immediately from results of section 6[4] taking into account that $Y_\eta \to Y$ is a regular morphism so that $\mathcal{O}_{X,x} \to \mathcal{O}_{Y\eta,y}$ should be regular. Taking into account 2.3, we obtain thus:

Corollary 2.9. *With the notations for 2.8, let Z be the set of $x \in X$ such that $P(\mathcal{F}_x)$ is false. Then in order for $\mathcal{G}_\eta$ to satisfy the condition P at all its points, it is necessary and sufficient for $f(Z)$ to be a finite subset of P, or to have* $\dim \overline{f(Z)} = 0$.

Indeed, 2.8 tells us that $h^{-1}(Z)$ is the P-singular subset of $\mathcal{G}_\eta$ and that it is empty if and only if $f(Z)$ is finite by 2.3 (N.B. h denotes the morphism $Y_\eta \to X$; I have just realized that the letter P in 2.8 has been used in two different ways).

Corollary 2.10. *Let $y \in Y_\eta$, in order that Y_η be regular, respectively satisfy the property R_k (reference) at y, respectively be normal at y, it is necessary*

[4]Interp.: clear up this reference. Is it EGA IV ?

and sufficient that X satisfy the same property at x. Let Z be the set of those points of X where X is not regular, resp. $\mathcal{E}_k$ (resp. normal); for Y_η to be regular, resp. to satisfy R_k, resp. normal, it is necessary and sufficient that $f(Z)$ be finite, i.e. $\dim \overline{f(Z)} = 0$.

Same proof as for 2.8 and 2.9. We must give the different references ensuring that Z be closed (as we must know that it is constructible to apply 2.3).

Let us point out that in 2.10 we do not talk at all about the corresponding geometric properties; the results described are of 'absolute' nature. We now examine the properties of geometric nature. (We could possibly take the opportunity to change the section.)

§3 Generic hyperplane section: geometric irreducibility and connectedness

Theorem 3.1 (Bertini-Zariski). *Assume X geometrically irreducible and* $\dim f(X) \geq 2$. *Then the generic hyperplane section Y_η has the same property.*

Let K/k be the function field of X and let $n = \dim P$; introducing the affine coordinates $T_1, \dots, T_n$ in P (by choosing a hyperplane at infinity H^∞ such that $f(X)$ is not contained in it) and $S_1, \dots, S_n$ the affine coordinates in $\check{P}$, we see that the function field L of Y_η can be identified with the field of fractions of the integral domain $K[S_1, \dots, S_n]/(\overline{\Sigma t_i S_i - 1})$ where the $t_i \in K$ are the images of T_i under $f: X \to P$. Since $\dim \overline{f(x)} > 0$, the t_i are not all algebraic over k, à fortiori they are not all zero; for example, take $t_n \neq 0$. Then, we realize immediately that we have $L = K(S_1, \dots, S_{n-1})$ (pure transcendental extension), $S_n \in L$ given by the equation $\Sigma t_i S_i - 1 = 0$ as a function of the S_i $(1 \leq i \leq n-1)$ and the t_i $(1 \leq i \leq n)$. On the other hand, $k' = k(\eta)$ can be identified with $k(S_1, \dots, S_n)$ and the canonical inclusion $k' \to L$ is obtained by sending S_i to S_i
i.e. k' being a subextension of L, is the subextension generated by the S_i $(1 \leq i \leq n)$ or what is evidently the same by the S_i $(1 \leq i \leq n-1)$ and by $S_n = a_0 + a_1 S_1 + \dots + a_{n-1} S_{n-1}$, where $a_0 = t_n^{-1}$, $a_i = -t_i t_n^{-1}$ for $1 \leq i \leq n-1$.

We notice that the field generated by the a_i and by the t_i is obviously the same, their common transcendence degree is nothing else but the dimension of $f(X)$.

(N.B. It would be appropriate to include this birational description at least as a corollary to 2.1). The proof of 3.1 is thus reduced to that of

Lemma 3.1.1 (Zariski). *(See interpreter's note at the end of section [Interp.]) Let k be a field, K an extension of finite type over k, m an integer ≥ 0, a_i $(0 \leq i \leq m)$ the elements of K such that the transcendence degree of*

$k(a_0, \ldots, a_m)$ over k is ≥ 2. Let $L = K(S_1, \ldots, S_m)$ and k' be the subfield $k' = k(S_1, \ldots, S_m,\ a_0 + \sum_1^m a_i S_i)$ of L (the S_i being indeterminates). If K is a primary extension of k then L is a primary extension of k'.[5]

This lemma, or lemmata that look like a brother, wander all over the literature. That is why I leave it up to you the choice of the place from where you will copy a proof, i.e. I do not feel inspired to find a proof with my own means.

Corollary 3.2. *Assume f is unramified or the characteristic of k is zero, and $\dim \overline{f(X)} \geq 2$. Then if X is geometrically integral, the same is true about Y_η.*

Indeed, geometrically integral = geometrically irreducible + separable.

Corollary 3.3. *Assume that k is algebraically closed and that for every irreducible component X_i of X we have $\dim f(X_i) \geq 2$. Moreover suppose that X is $\mathfrak{S}$-connected, where $\mathfrak{S}$ is the set of closed subsets Z of X such that $\dim \overline{f(Z)} = 0$ (i.e. for every such Z, $X - Z$ is connected). Under such conditions, Y_η is geometrically connected over $k(\eta)$.*

Indeed, by a lemma that ought to appear in sect. 6[6] with Hartshorne's theorem, the hypothesis means that we can join any two irreducible components X' and X'' of X by a chain of irreducible components $X_0 = X', \ldots, X_n = X''$ such that two consecutive ones have an intersection not in $\mathfrak{S}$ then the inverse images X'_η and X''_η are joined by a chain of $X_{i\eta}$ which are geometrically connected over $k(\eta)$ by 3.1 and the intersection of two consecutive ones is not empty by 2.3.

It follows (since $Y_\eta = X_\eta$ is the union of the $X_{i\eta}$, X_i running through the set of irreducible components of X) that Y_η is geometrically connected over $k(\eta)$. q.e.d.

Interpreters's note to 3.1.1: This should be compared with Zariski's collected papers (MIT Press) vol. 1, page 174, vol. 2, page 304. Also Zariski-Samuel vol. 1, page 196, vol. 2, page 230 of the GTM Springer edition. Also Jouanolou: Théorème de Bertini et applications, Th. 3.6 and Section 6.

[5] primary extension probably means that the smaller field is algebraically closed in the larger one (or quasi algebraically closed) [Interp.]. Jouanolou Thm. 3.6 [Interp.]

[6] Ask A.G.

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