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The Effect of Truncation on the UMVUE of the Black-Scholes Option Price*

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Abstract

The uniform minimum variance unbiased estimator (UMVUE) of the Black-Scholes option price is an infinite power series in the sample variance rate. We study the effect of truncation when only a finite number of terms in this series are used to estimate the Black-Scholes option price. We use the market data to compare the performance of the classical biased estimator and the UMVUE. Our empirical results indicate that the UMVUE performs no better than the classical estimator.

1. Introduction

The estimation of an American call option price continues to occupy a great deal of attention of the Academics as well as the market practitioners. We refer to Smith (1976) for a detailed review. For more recent work we refer to Barone-Adesi and Whaley (1987), Brown and Kritzman (1990), Cox and Rubinstein (1985), Geske (1984), and Jarrow and Rudd (1983). The Black-Scholes (1973) model was developed to value a European option as a function of the true variance rate. It assumes absence of cash dividends. In order to apply the model to American options, it is adjusted for known dividends. See, for example, Jarrow and Rudd (1983), p. 124.

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The true variance rate in the Black-Scholes formula is unobservable. An unbiased estimator of the variance is frequently substituted in its place. The resulting estimator is a biased estimator of the option price. An unbiased estimator of the option price can be obtained by expanding the Black-Scholes option price into a Taylor series expansion in powers of the true variance rate, and then estimating each term by its uniform minimum variance unbiased estimator (UMVUE).

In this paper we use the market data to compare the performance of the classical biased estimator of the Black-Scholes option price and the UMVUE. Our results indicate that the UMVUE suffers from the same drawbacks as the classical estimator. In fact, we found that the difference in magnitude between the classical estimator and the UMVUE was never more than 1.5 cents.

2. Overview

We will use the following notation.

$$\begin{aligned} c &= \text{ option exercise price,} \\ x &= \text{ current stock price,} \\ r &= \text{ riskless (daily) rate of return,} \\ T &= \text{ option time (days) to maturity,} \\ v^2 &= \text{ true (daily) variance rate of the stock,} \\ g &= (x/c)\exp(rT), \\ d_1 &= [\ell n\, g + (1/2)v^2 T]/\left(v\sqrt{T}\right), \\ d_2 &= d_1 - v\sqrt{T}, \quad \text{and} \\ \Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt, \text{standard normal distribution function.} \end{aligned}$$

Then the Black-Scholes option price is given by

$$\begin{aligned} h(v^2) &= x\Phi(d_1) - c\exp(-rT)\Phi(d_2) \\ &= x\{\Phi(d_1) - (1/g)\Phi(d_2)\}. \end{aligned} \tag{2.1}$$

The best unbiased estimator of v^2 using closing prices is known to be $S^2 = (1/n)\sum_{t=1}^{n}[(Y_t - Y_{t-1})^2/Y_t^2]$, where Y_t is the stock price on day t and n is the number of trading days. The usual estimator of $h(v^2)$ is, therefore, the substitution principle estimator (known as the classical estimator)

$$\tilde{h} = h(S^2) = x\{\Phi(x_1) - (1/g)\Phi(x_2)\} \tag{2.2}$$

where x_1, x_2 are obtained from d_1, d_2 by substituting S^2 for v^2. However, since h is a nonlinear function of S^2, $\tilde{h}$ is not unbiased for $h(v^2)$.

Recently Butler and Schachter (1986) obtained the uniformly minimum variance unbiased (UMVU) estimator of $h(v^2)$ by expanding ϕ into a power series and estimating each term by its UMVU estimator. For completeness and ease of reference, we give below a brief outline of their derivation and the corrected expression of the UMVU estimator.

Consider the power series expansion of $\Phi(x)$ due to Laplace (1875): for $x \geq 0$

$$\begin{aligned}\Phi(x) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)2^k k!} \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \sum_{k=1}^{N} \frac{(-1)^k x^{2k+1}}{(2k+1)2^k k!}. \end{aligned} \tag{2.3}$$

It is well known that the series converges slowly *except when x is small* (Kendall and Stuart (1958), p. 136). If the summation is stopped when $k = N$, and the remainder is $R_N(x)$, then

$$|R_N(x)| < x^{2N+3} \exp(x^2/2) / \left[2^{N+1}(2N+3)\sqrt{2\pi}\right]. \tag{2.4}$$

In view of (2.1) and (2.3), we can rewrite

$$\begin{aligned} &h(v^2) = x\{\Phi(d_1) - (1/g)\Phi(d_2)\} \\ &= x\left\{\frac{1}{2}\left(1 - \frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)2^k k!}\left[d_1^{2k+1} - \frac{1}{g} d_2^{2k+1}\right]\right\} \\ &= x\left\{\frac{1}{2}\left(1 - \frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1^k)}{(2k+1)2^k k!}\left[d_1^{2k+1} - \frac{1}{g} d_2^{2k+1}\right]\right\}. \end{aligned} \tag{2.5}$$

Substituting for d_1 and d_2, expanding d_1^{2k+1} and d_2^{2k+1} and simplifying, we get

$$\begin{aligned} h(v^2) = x\Bigg\{&\frac{1}{2}\left(1 - \frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)2^k k!} \\ &\cdot \sum_{j=0}^{2k+1} \binom{2k+1}{j} (\ell n\, g)^j \left(\frac{1}{2}\right)^{2k+1-j} T^{(2k+1)/2-j} \\ &\cdot v^{2k+1-2j}\left[1 + (-1)^{2k+2-j} \cdot \frac{1}{g}\right]\Bigg\}. \end{aligned} \tag{2.6}$$

This is a power series in v and, therefore, we can use the linearity property of the expectation operator to get an unbiased estimator (in fact, the uniformly minimum variance unbiased estimator, see Butler and Schachter (1986), p.

349) of $h(v^2)$ by replacing each term in the double sum on the right hand side of (2.6) by its unbiased estimator.

Recall that S^2 has a gamma $(n/2, n/(2v^2))$ distribution. Using properties of gamma function (see, for example, Rohatgi (1984), p. 398), we see immediately that for $\ell + n > 0$

$$\begin{aligned}\mathcal{E}S^v &= \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{n}{2v^2}\right)^{n/2}\int_0^\infty (s^2)^{\frac{n+\ell}{2}-1}\exp\left(-\frac{n}{2v^2}s^2\right)ds^2 \\ &= \frac{\Gamma\left(\frac{n+\ell}{2}\right)}{\Gamma(\frac{n}{2})}\left(\frac{2v^2}{n}\right)^{\ell/2}. \end{aligned} \tag{2.7}$$

It follows that for $\ell > -n$

$$\left(\frac{n}{2}\right)^{\ell/2}\frac{\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n+\ell}{2}\right)}S^\ell \tag{2.8}$$

is the (uniformly minumum variance) unbiased estimator of v^ℓ. We note that expression (2.7) differs from expression (15) in Butler and Schachter (1986) in that the term (in their notation) $k^{-\ell/2}$ is missing from their expression (15) and also in their expression (A.7). This leads them to the erroneous conclusion (on p. 356) that the coefficient of S^ℓ in the unbiased estimator of v^ℓ "becomes quite large rather quickly." Indeed, as $n \to \infty$,

$$c_{n,\ell} = \left(\frac{n}{2}\right)^{\ell/2}\frac{\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n+\ell}{2}\right)} \longrightarrow 1$$

so that the unbiased estimator in (2.6) is consistent for v^ℓ.

It should also be noted that the comments on top of page 349 of Butler and Schachter (1986) about gamma function and $\mathcal{E}(S^2)^j$ are also incorrect. Indeed, $\mathcal{E}S^\ell$ exists for negative real numbers ℓ provided $\ell > -n$.

Replacing $v^{2k+1-2j}$ in (2.6) by its unbiased estimator obtained by taking $\ell = 2k+1-2j$ in (2.8), we see that

$$\begin{aligned}\hat{h}(v^2) = x\Bigg\{&\frac{1}{2}\left(1-\frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)2^k k!}\sum_{j=0}^{2k+1}\binom{2k+1}{j}(\ell n\, g)^j \\ &\cdot\left(\frac{1}{2}\right)^{2k+1-j} T^{(2k+1)/2-j}\left[1+(-1)^{2k+1-j}\cdot\frac{1}{g}\right]\left(\frac{n}{2}\right)^{\frac{2k+1}{2}-j} \\ &\cdot\frac{\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n+2k+1-2j}{2}\right)}S^{2k+1-2j}\Bigg\}\end{aligned} \tag{2.9}$$

is the uniformly minimum variance unbiased estimator of $h(v^2)$. For convenience, let us write

$$\hat{h}(v^2) = \lim_{N\to\infty}\hat{h}_N \tag{2.10}$$

where

$$\hat{h}_N = x\Big\{\frac{1}{2}\left(1-\frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)2^k k!} \sum_{j=0}^{2k+1} \binom{2k+1}{j} (\ell n\, g)^j$$
$$\cdot \left(\frac{1}{2}\right)^{2k+1-j} T^{(2k+1)/2-j} \left[1 + (-1)^{2k+2-j} \cdot \frac{1}{g}\right] \left(\frac{n}{2}\right)^{(2k+1)/2-j}$$
$$\cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2k+1-2j}{2}\right)} S^{2k+1-2j}\Big\}. \tag{2.11}$$

In practice we cannot use the estimator (2.9) which is based on an infinite number of terms, but only $\hat{h}_N$ which is based on a finite number of terms. Unfortunately, this destroys the unbiasedness property of the estimator. Indeed, $\hat{h}_N$ is not unbiased for $h(v^2)$, but only asymptotically unbiased. However, we can reduce the bias by taking N sufficiently large.

We emphasize that the comptuation of $\hat{h}_N$ can be carried out recursively. Indeed, from (2.11) we have

$$\hat{h}_N = \hat{h}_{N-1} + \frac{x}{\sqrt{2\pi}} \frac{(-1)^N}{(2N+1)2^N N!} \sum_{j=0}^{2N+1} \binom{2N+1}{j} (\ell n\, g)^j$$
$$\times \left(\frac{1}{2}\right)^{2N+1-j} T^{(2N+1)/2-j} \left[1 + (-1)^{2N+2-j} \cdot \frac{1}{g}\right] \left(\frac{n}{2}\right)^{(2N+1)/2-j}$$
$$\times \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2N+1-2j}{2}\right)} S^{2N+1-2j}, \tag{2.12}$$

for $N = 1, 2, \ldots$ where

$$\hat{h}_0 = x\Big\{\frac{1}{2}\left(1-\frac{1}{g}\right) + \frac{1}{\sqrt{2\pi}}\Big[\frac{1}{2}\left(1+\frac{1}{g}\right)\left(\left(\frac{Tn}{2}\right)^{1/2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right) S$$
$$+ (\ell n\, g)\left(1-\frac{1}{g}\right)\left(\frac{Tn}{2}\right)^{-1/2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} S^{-1}\Big]\Big\}. \tag{2.13}$$

Butler and Schachter (1986) check the bias of $\hat{h}_N$ by setting $N \equiv 40$ assuming that Black-Scholes is the correct formula for option price. But, unfortunately, there was no attempt to compare the performance of $\hat{h}_{40}$ and $\tilde{h}$ using empirical data. This is done in Section 3.

3. The Sample and the Methodology

For the purposes of this study, attention was restricted to twenty-eight of the thirty stocks that comprised the Dow-Jones Industrial Average on January 3, 1989. Navistar was excluded because there were no options on it and Primerica, because the *Wall Street Journal* did not report any trade on that day. The sample consists of 199 options traded on January 3, 1989. The call prices were taken from the January 4, 1989 issue of the WSJ. For the comptuation of S^2, the unbiased estimator of v^2, we used the closing prices of the corresponding stock for the preceding 90 day period. Our computations took account of anticipated dividend payments. For this purpose, we used the dividend adjustment technique desribed on p. 124 of Jarrow and Rudd (1983).

In the computation of $\tilde{h}$ and $\hat{h}_{40}$, we took T to be the total number of days until maturity and $r = 7$ percent per annum. For each call we computed $\tilde{h}$ and $\hat{h}_{40}$ using (2.2) and (2.12), respectively, and then computed the bias found between the estimated model price and the market price. The number of terms used in computing $\hat{h}_{40}$ equalled the number of terms needed before the difference between two successive terms of $\hat{h}_{40}$ became less than 10^{-4}.

4. Conclusions

Our computitons lead us to the following conclusions.

(i) Neither $\tilde{h}$ nor $\hat{h}_N$ predicts the market call option price very well. Their performance is particularly poor for out-of-money calls.

(ii) Both estimators showed negative bias in 70 percent of the cases. Table 1 summarizes the results on average bias by time to maturity and strike price both for in-the-money and out-of-money options. The magnitude of the bias appears insignificant for options with strike price between 11 and 40 dollars, and between 81 and 100 dollars. There does not appear to be any other pattern in the magnitude or the direction of the bias.

(iii) The UMVUE exceeded the Black-Scholes estimate in about 65 percent of the cases. In general, the magnitude of the difference between the two estimates was small. In fact, the absolute value of the difference was less than one cent in all but four cases. This explains why the unbiased estimator does no better than the classical estimator.

(iv) Similar computations were made with similar results for the case when $n = 60$ days was used in computing s^2.

In summary, we tested the market performance of the unbiased estimator as well as the classical estimator on the DJIA stocks. We found that the unbiased estimator performs no better than the classical biased estimator. In fact, the two estimators give the same estimate of the option price for all practical purposes.

Table 1: Average Bias

	Number of Options	Average Bias (Dollars) Call less Market	B/S less Market
1. All Options: By Time to Maturity			
1-30 days	70	-0.1007	-0.1010
31-60 days	57	-0.1516	-0.1538
61-90 days	16	0.0249	0.0209
91-120 days	45	-0.1663	-0.1706
121-150 days	11	-0.1430	-0.1480
2. All Options: By Strike Force			
\$11 - \$20	5	-0.0968	-0.0967
\$21 - \$40	52	-0.0384	-0.0392
\$41 - \$60	88	-0.1205	-0.1229
\$61 - \$80	16	-0.4600	-0.4618
\$81 - \$100	23	-0.0634	-0.0677
\$101 - \$120	12	-0.2375	-0.2420
\$121 - \$140	3	0.1338	0.1257
3. In the Money Options			
a) By time to maturity			
1-30 days	39	-0.1004	-0.1008
31-60 days	25	-0.2433	-0.2455
61-90 days	8	-0.0938	-0.0964
91-120 days	23	-0.3089	-0.3117
121-150 days	2	-0.2816	-0.2827
b) By strike price			
\$11 - \$20	5	-0.0968	-0.0967
\$21 - \$40	33	-0.1137	-0.1138
\$41 - \$60	33	-0.1563	-0.1590
\$61 - \$80	11	-0.5567	-0.5585
\$81 - \$100	8	0.0933	0.0882
\$101 - \$120	7	-0.5201	-0.5215
4. Out of the Money Options			
a) By time to maturity			
1-30 days	31	-0.1011	-0.1012
31-60 days	32	-0.0800	-0.0822
61-90 days	8	0.1436	0.1382
91-120 days	22	-0.0173	-0.0231
121-150 days	9	-0.1122	-0.1180
b) By strike price			
\$21 - \$40	19	0.0925	0.0903
\$41 - \$60	55	-0.0990	-0.1012
\$61 - \$80	5	-0.2474	-0.2490
\$81 - \$100	15	-0.1470	-0.1509
\$101 - \$120	5	0.1582	0.1493
\$121 - \$140	3	0.1338	0.1257

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