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Duality in Representation Theory

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Abstract

The Schur-Weyl Duality Theorem motivates the general definition of dual representations. Such representations arise in a very large class of groups, including all compact, nilpotent and semidirect product groups, as well as the complementary groups in the physics literature and the reductive dual pairs in the mathematics literature. Several applications of the notion of duality are given, including the determination of polynomials and operator invariants, and the discovery of unusual dual algebras.

Introduction

In 1901 I. Schur proved a theorem [Sc] that was later expanded by Weyl in a form [We] that is now called the Schur-Weyl Duality Theorem. This theorem, which is briefly reviewed in the next section, relates the representation theory of two groups, the general linear group and the symmetric or permutation group. It is the first theorem, as far as we know, that classifies, up to isomorphism, all irreducible representations of one group in terms of all irreducible rational representations of the other. As we shall show such a phenomenon in which representations of pairs of groups are related turns out to be quite general. It includes the complementary pairs of groups introduced in the physics literature by Moshinsky and Quesne [MQ] and the reductive dual pairs introduced in the mathematics literature by many mathematicians [Ge, GK, KV, Sa, . . .], and especially Howe [Ho]. All compact groups can be written as dual representations and it seems likely that all nilpotent and semidirect product groups can be viewed as dual representations although this remains to be proven.

There are many applications of the theory of dual representations. First, knowing that the representations of two groups are dual to one another allows one to get the algebras generated by the group actions, from which information can be obtained about the commutants. One rather surprising connection of such dual algebras is with the theory of polynomial invariants.

Second, given two groups G and G' whose representations are dual, the restriction of the representation of G to a subgroup H may give rise to an extension of the representation of G' to a group H' containing G'. Such a "see-saw" phenomenon (this term was introduced by Kudla) wherein H' is dual to H gives considerable insight on the nature of the restriction of G to H or H' to G'. As will be shown, it is possible to view the decomposition of tensor products in this context and use the dual pair structure to resolve the multiplicity problem.

Finally, giving a representation of a group along with its spectral decomposition can lead to unusual representations of a dual group, or even to a dual algebra whose structure might not be known or conveniently characterized. In the next section we will give a general definition of dual representations and provide a number of examples to illustrate the generality of the definition. The third section will then give a number of applications of the notion of duality.

§2. Definitions and Examples of Dual Representations

To motivate the definition of dual representations we will formulate the Schur-Weyl Duality Theorem in a somewhat unfamiliar way, so that it can easily be generalized. Let $P^{\underbrace{(1,\cdots,1)}_{n}}(\mathbb{C}^{n\times N})$ denote the space of all polynomial functions F on $\mathbb{C}^{n\times N}$ which satisfy the covariant condition

$$F(dZ) = \det(d)F(Z) \ , \ \forall\ d = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix}, \ Z \in \mathbb{C}^{n\times N}.$$

Let S_n denote the symmetric group of degree n and $GL(N,\mathbb{C})$ denote the complex general linear group of order N. A joint action of $S_n \times GL(N,\mathbb{C})$ on $P^{(1,\cdots,1)}$ is defined by

$$[L(\sigma)\otimes R(g)]F(Z) = F(\sigma^{-1}Zg)$$

$$\forall\ F \in P^{(1,\cdots,1)},\ \forall\ \sigma \in S_n,\ \forall\ g \in GL(N,\mathbb{C}),\ \text{and}\ \forall\ Z \in \mathbb{C}^{n\times N}$$

Then the Schur-Weyl Duality Theorem can be stated as follows:

3) Let $Z \in \mathbb{C}^{n\times N}$ and $d\mu(Z) = (1/\pi^{nN})\exp(-tr(ZZ^t))dZ$ and set

$$\begin{aligned}\mathcal{H} &= \mathcal{F}(\mathbb{C}^{n\times N})\\ &= \{f : \mathbb{C}^{n\times N} \longrightarrow \mathbb{C} : f \text{ entire and } \int_{\mathbb{C}^{n\times N}} |f(Z)|^2\, d\mu(Z) < \infty\}\end{aligned}$$

Let $G = GL(N,\mathbb{C})$ and $G' = GL(n,\mathbb{C})$ and define

$$\begin{aligned}[L(g')f](Z) &= f(g'^{-1}Z)\\ [R(g)f](Z) &= f(Zg)\end{aligned}$$

Then $\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_\lambda \oplus I_\lambda$ and the representation $L \otimes R|_{I_\lambda}$ is irreducible, where λ denotes both a representation of G and G' in the class of irreducible representation of G (and G') labeled by an r-tuple of integers $(\ell_1, \dots, \ell_r)$ satisfying $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r \geq 0$, $r = \min(n, N)$.

4) Generalization of Example No. 3. Let $\mathcal{H}$ be the same as in Ex. 3. Let $G = \underbrace{GL(N,\mathbb{C}) \times \cdots \times GL(N,\mathbb{C})}_{m \text{ copies}}$. Let $p_1, \dots, p_m$ be integers such that $p_1 + p_2 + \cdots + p_m = n$, and let

$$G' = \underbrace{GL(p_1,\mathbb{C}) \times \cdots \times GL(p_m,\mathbb{C})}_{m}$$

Partition $Z \in \mathbb{C}^{n\times N}$ in block form as

$$Z = \begin{bmatrix} Z_1 \\ \vdots Z_m \end{bmatrix} \quad \text{with } Z_i \in \mathbb{C}^{p_i\times N} \quad 1 \leq i \leq m;$$

then G acts on $\mathbb{C}^{n\times N}$ via

$$Z \cdot (g_1, \dots, g_m) = \begin{bmatrix} Z_1 g_1 \\ \vdots \\ Z_m g_m \end{bmatrix} \quad \forall\, g_i \in GL(N,\mathbb{C})$$

and G' acts on $\mathbb{C}^{n\times N}$ via

$$(k_1, \dots, k_m) \cdot Z = \begin{bmatrix} k_1^{-1} Z_1 \\ \vdots \\ k_m^{-1} Z_m \end{bmatrix}, \quad k_i \in GL(p_i).$$

These actions induce a representation of $G' \times G$ on $\mathcal{H}$ defined by

$$\begin{aligned}&[(L \otimes R)((k_1, \dots, k_m), (g_1, \dots, g_m))]f(Z)\\ &\qquad = f[(k_1, \dots, k_m) \cdot Z \cdot (g_1, \dots, g_m)] \quad \forall\, f \in H.\end{aligned}$$

Theorem 2.4. *The representation $L \otimes R$ of $G' \times G$ on $\mathcal{H}$ is decomposed into irreducible subrepresentations labeled by the double signature $(\mu_G, \mu_{G'})$ where $\mu_{G'}$ is an irreducible representation of G' indexed by an n-tuple of integers of the form*

$$\left(M_1^{(1)}, \ldots, M_{p_1}^{(1)}; M_1^{(2)}, \ldots, M_1^{(m)}, \ldots, M_{p_m}^{(m)}\right)$$

with $M_1^{(i)} \geq M_2^{(i)} \geq \cdots \geq M_{p_i}^{(i)} \geq 0$, $1 \leq i \leq m$ and μ_G is a tensor product of irreducible representations of G indexed by the same n-tuple of integers. Moreover, each irreducible subspace of this orthogonal direct sum decomposition of $\mathcal{H}$ is an isotypic component of both μ_G and $\mu_{G'}$.
See [KT1] and [KT3].

Example 4 is a typical example of "dual (or complementary) pairs" as defined by M. Moshinksy [MQ] and generalized by R. Howe [Ho]. An example of the "see-saw" phenomenon arises if the subgroup H of G is chosen to be $SO(N)$; the dual to H is $H' = Sp(2m, \mathbb{R})$ acting on $\mathcal{H}$ as in Example 3 with $n = 2m$. Note that the irreducible representations of $Sp(2m, \mathbb{R})$ are infinite-dimensional metaplectic representations. (See [KV], [GK] and [MQ]).

§3. Applications of Dual Representations

1) *The invariant theory of block diagonal subgroups of $GL(n, \mathbb{C})$.* Let $G' = GL(p_1, \mathbb{C}) \times \cdots \times GL(p_m, \mathbb{C})$ and $\mathcal{H}$ be the same as in Example 4 of Section 2. Let $n = p_1 + \cdots + p_m$ and consider the algebra $S(\mathbb{C}^{n \times n})$ of all polynomial functions on $\mathbb{C}^{n \times n}$. Then the adjoint representation of $GL(n, \mathbb{C})$ gives rise to the coadjoint representation of $GL(n, \mathbb{C})$ on $S(\mathbb{C}^{n \times n})$. The polynomials of $S(\mathbb{C}^{n \times n})$ that are fixed by the restriction of this coadjoint representation to the block diagonal subgroup G' of $GL(n, \mathbb{C})$ form a subalgebra of $S(\mathbb{C}^{n \times n})$ called the subalgebra of G'-invariants. Using the work of Procesi [Pr] on the dual G of G' and a Frobenius reciprocity theorem relating the dual action of G and G' on $\mathcal{H}$ gives the following.

Theorem 3.1. *(See [KT4]) If a matrix $X \in \mathbb{C}^{n \times n}$ is partitioned in block form as*

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \vdots \\ X_{m1} & \cdots & X_{mm} \end{bmatrix}$$

where each X_{ij} is a $p_i \times p_j$ matrix, $1 \leq i, j \leq m$, then the subalgebra of all G'-invariant polynomials is finitely generated by the constants and the functions of the form

$$\mathit{Trace}(X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_q i_1})$$

for all $i_k = 1, \dots, m$*;* $k = 1, \dots, q$*.*

2) *The resolution of the multiplicity in the decomposition of tensor products of group representations.* Example 4 and Theorem 2.4 are typical of the way the "see-saw" phenomenon is used. In this example, if we consider the restriction of the representation R of $G = \underbrace{GL(N,\mathbb{C}) \times \cdots \times GL(N,\mathbb{C})}_{m}$ to the diagonal subgroup $H = \{(g, \dots, g) : g \in GL(N,\mathbb{C})\}$, then we have the problem of decomposing an m-fold tensor product $M^{(1)} \otimes \cdots \otimes M^{(m)}$ of irreducible representations of $GL(N,\mathbb{C})$, where each $M^{(i)}$, $1 \le i \le m$, is the signature of an irreducible representation of $GL(N,\mathbb{C})$. In general, the spectral decomposition of this tensor product involves multiplicity. To resolve this multiplicity H-invariant differential operators (or generalized Casimir operators) are introduced and their eigenvalues are used to label different copies of the same irreducible representation of H occurring in this decomposition. These H-invariant differential operators are related to the polynomial invariants in Application 1 (see [KT1] and [KT2]).

This leads to the following general setup. Let G be any reductive Lie group with the holomorphic representation (on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ of Example 3) defined by $[R(g)f](Z) = f(Z \cdot g)$, and consider a holomorphic irreducible representation χ of G on a subspace V_G^χ of $\mathcal{F}(\mathbb{C}^{n\times N})$. If there is a dual action of G' on a subspace $V'^{\chi}_{G'}$ such that the isotypic component is $V'^{\chi}_{G'} \otimes V_G^\chi$ (in general, the action of G' is much larger than just the left action unless $G' = GL(n,\mathbb{C})$ and often one only knows the Lie algebra action of G') then the action of G' may be used to resolve the problem of the restriction of the representation χ of G to a subgroup in the following sense. Consider a subgroup H of G with an irreducible representation μ on W_H^μ. Since H is contained in G, its dual H' contains G' and has a representation of W'^{μ}_H such that the isotypic component of the dual representation in $\mathcal{F}(\mathbb{C}^{n\times N})$ is $W'^{\mu}_{H'} \otimes W_H^\mu$. We then have the generalized Frobenius reciprocity theorem:

The multiplicity $\dim Hom(V'^{\chi}_{G'}, W'^{\mu}_{G'})$ *of the irreducible representation* $V'^{\chi}_{G'}$ *in the restriction of* $W'^{\mu}_{H'}$ *to* G' *is equal to the multiplicity dim Hom* (V_H^μ, V_H^χ) *of the irreducible representation* μ *of* H *in the restriction of* W_G^χ *to* H*. If* $\dim(V_H^\mu, V_H^\chi) > 1$*, then generalized Casimir operators* defined in the same way as above can be used to "break" this multiplicity.

3) *Construction of bases for irreducible representation spaces.* Dual representation theory can also be used to find bases for simple G-modules. We started with the simple example of Gelfand-Žetlin bases for $GL(n,\mathbb{C})$ by considering the problem of decomposing tensor products of representations of $GL(N,\mathbb{C})$ in Example 4 of Section 2. In Theorem 2.4 if the tensor products μ of irreducible representations of $G = \underbrace{GL(N,\mathbb{C}) \times \cdots \times GL(N,\mathbb{C})}_{m}$ consist only of signatures of the form $M^{(i)} = \underbrace{(M_1^i, 0, \dots, 0)}_{N}$ for all $i =$

$1, \ldots, m$, then using a formula of Weyl the spectral decomposition of the tensor product of this particular $M^{(i)}$ with a general signature is multiplicity free. Thus if we consider the coupling scheme

$$\left(\cdots\left(\left(M^{(1)} \otimes M^{(2)}\right) \otimes M^{(3)}\right) \cdots M^{(m)}\right)$$

then in the dual representation $G' = GL(n, \mathbb{C})$ we have a corresponding chain of subgroup representations $GL(1, \mathbb{C}) \subset GL(2, \mathbb{C}) \subset \cdots \subset GL(n, \mathbb{C})$ which is precisely the Gelfand-Žetlin chain. In [KT1] this construction is carried out completely to find Gelfand-Žetlin bases for irreducible representation spaces of $GL(n, \mathbb{C})$. If different coupling schemes are used, then one can construct different types of bases. A similar construction for the Gelfand-Žetlin bases for the chain $SO(2) \subset SO(3) \subset \cdots \subset SO(N)$ can be carried out as follows. It is known that successive restrictions of an irreducible representation of the chain of subgroups $SO(N) \supset SO(N-1) \supset \cdots \supset SO(2)$ is decomposed into multiplicity-free subrepresentations, but the difficulty is that odd and even orthogonal groups have a very different structure. So it is natural to consider the chain $SO(N) \supset SO(N-2) \supset SO(N-4) \supset \cdots$; set $G = SO(N)$, $H = SO(N-2)$ and let $V_G^{(M)}$ be an irreducible G-module and $W_H^{(m)}$ be an irreducible H-module. It is easy to compute the multiplicity of $W_H^{(m)}$ in the restriction $V_H^{(M)}$ of G to H. Now the dual G'-module is the metaplectic representation of $Sp(2n, \mathbb{R})$, so we need to calculate the dual H'-module whose raising operators will send $W_H^{(m)}$ into the intersection of the isotypic component $I^{(M)}$ with the isotypic component of $I^{(m)}$. Again there is multiplicity which can be resolved with generalized $SO(N-1)$ Casimir operators. A similar construction also can be carried out for the compact symplectic groups; further the methods used to decompose tensor products of the $GL(N, \mathbb{C})$ groups can also be applied to the orthogonal and symplectic groups.

4) *Construction of new representations and algebras via duality.* Using the notion of dual representation can lead to representations of various groups or algebras which are generally difficult to classify. The procedure can be described as follows. Let K be a compact group and let ρ be a finite-dimensional representation of degree N of K on a complex vector space V. The many-particle symmetric Fock space $\mathfrak{L}(V)$ is defined as

$$\mathfrak{L}(V) = \sum_{n=0}^{\infty} \oplus \underbrace{(V \otimes \cdots \otimes V)}_{n} \text{sym}$$

where $\underbrace{(V \otimes \cdots \otimes V)}_{n}$ sym is the n-fold symmetrized tensor product of V. If $\{\hat{e}_1, \ldots, \hat{e}_N\}$ is an orthonormal basis for V, then $\hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_n}|_{\text{sym}}$ is an orthogonal basis of the n-particle subspace of $\mathfrak{L}(V)$. The correspondence between $\mathcal{L}(V)$ and the space $\mathcal{F}(\mathbb{C}^{1\times N})$ given in Example 4 of Section 2 is given by

$$\hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_n}|_{\text{sym}} \longrightarrow Z_{i_1} \ldots Z_{i_n}$$

so that an action of K on $\mathcal{F}(\mathbb{C}^{1\times N})$ is given by $R(k)f(Z) = f(ZD(k))$, $\forall\, f \in \mathcal{F}(\mathbb{C}^{1\times N})$, where $D(k)$ is the matrix of $\rho(k)$ relative to the orthogonal basis $\{\hat{e}_1, \ldots, \hat{e}_N\}$. The algebra A_V^K of operators on $\mathcal{F}(\mathbb{C}^{1\times N})$ which commute with $D(k)$, $k \in K$, is in general an infinite-dimensional algebra which has a Cartan-Weyl structure, with diagonal, raising and lowering operators. This algebra is dual to the algebra generated by the operators $D(k)$, $k \in K$ and its representations are indexed by the representations of K appearing in the symmetric tensor products. The explicit construction of several algebras A_V^K is given in [K1] and [K2].

§4. Conclusion

In this article we have given a number of examples and applications of the notion of dual representations. We have however only scratched the surface of what we think is a very rich structure encompassing a very large class of group representations and related subjects. This makes us appreciate even more the simple idea initiated by Schur and Weyl.

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