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Tangential Linear Representations of Automorphism Groups of Free Groups

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Abstract

The automorphism group of a group acts morphically on its varieties of representations. The stabilizer of a representation acts on the tangent space of that representation; this provides a representation of the stablizer. An interpretation of D.D. Long's theorem that the Burau and Gassner representations of the braid group and pure braid group occur as tangential representations in the variety of two dimensional representations of a free group is given. For the full automorphism group of the free group, it is shown that the most faithful tangential representation factors through the second commutator quotient group.

Introduction

The set $R_n\Gamma = Hom(\Gamma, GL_n\mathbf{C})$ of n-dimensional complex representations of the finitely generated group Γ forms a complex algebraic variety: if Γ has generators $x_1, \ldots, x_d$, then $R_n\Gamma$ is the subset of $GL_n\mathbf{C}^{(d)}$ of d-tuples of matrices satisfying the relations of Γ, the correspondence sending the representation ρ to the tuple $(\rho(x_1), \ldots, \rho(x_d))$. In fact, this correspondence is clearly a closed immersion. The automorphism group $\mathrm{Aut}(\Gamma)$ of Γ acts on the representations $R_n\Gamma$ as algebraic variety automorphisms, so a subgroup H of $\mathrm{Aut}(\Gamma)$ which fixes a representation ρ will act on the Zariski tangent space $T_\rho(R_n\Gamma)$ to $R_n\Gamma$ at ρ. We will call these *tangential* representations.

Recently, Long [2, Thm. 2.4 and 2.6] has shown that, the Burau and Gassner representations of the braid group and pure braid group occur as (parameterized families of) tangential representations. The braid group and pure braid group on d strands are automorphism groups of the d-generator

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free group F_d. Long finds his representations in the closed analytic subvariety of R_2F_d of representations into $SU_2(\mathbf{C})$.

The Burau and Gassner representations are traditionally constructed by means of the Fox calculus, as explained, for example, by Birman [1, Examples 3 and 4]. The Fox calculus can be regarded as a method for calculating with free group cocycles (see, for example [3, p.59]), and it is a simple matter, recalled for completeness below, to see that the Fox calculus chain rule [1, Prop. 3.3, p.105] shows that the Fox calculus construction of representations of subgroups of $\mathrm{Aut}(F_d)$ is the same as the representations of those subgroups obtained on cocycle spaces $Z^1(F_d, M)$, where M is a module invariant under the relevant automorphism subgroup. We will call these *cocycle* representations.

As will be shown below, cocycle and tangential representations are closely allied: the tangent spaces of representation varieties are embedded in cocycle spaces [3, Prop. 2.2, p.33] and for the free group F_d we have an equality $T_\rho(R_nF_d) = Z^1(F_d, \mathrm{Ad} \circ \rho)$. (Here "Ad" is the adjoint representation of $GL_n\mathbf{C}$ on $M_n\mathbf{C}$ by conjugation.) In [3, Thm. 3.8, p.62], we showed that the differential of a map on representation varieties of free groups induced from a group homomorphism is given, under the above identification, by the Fox Jacobian matrix of the images of the generators. This result enables one to view the tangential representations as special cases of cocycle representations.

The purpose of this paper is to carry out the details of this connection. For this to be useful, one needs a reasonable supply of invariant representations. Here it is most convenient to consider closed subvarieties of invariant representations (this leads to parameterized families of tangential representations, as in Long's work) although an algebraic trick with generic representations allows these to be replaced by a single representation.

These methods suffice to obtain the Burau and Gassner representations as tangential representations; more precisely as direct factors of tangential representations. This works because the relevant cocycle representation embeds in the tangential one. We are then led to consider the possible representations of large subgroups of $\mathrm{Aut}(F_d)$. Unfortunately, it turns out that for subgroups large enough to contain the inner automorphism group $\mathrm{Inn}(F_d)$ the best that can be hoped for is a representation of the "induced I-A automorphism group F_d/F_d''" (terminology is from [1, Example 2, p.117] and means automorphisms of F_d/F_d'' induced from $\mathrm{Aut}(F_d)$, F_d'' is the second commutator group). We do find a faithful tangential representation here, however.

We use the basic results of the theory of representation varieties of [3], and adopt its notation. For Fox calculus and braid groups, our reference is [1] (see also [4]).

Tangential and Cocycle Representations

As noted, automorphisms of the free group $F_d = \langle x_1, \dots x_d \rangle$ will act on its representation varieties: if $\rho \in R_n F_d$ and $\alpha \in \mathrm{Aut}(F_d)$, then $\alpha\rho$ will be the representation $\rho \circ \alpha^{-1}$. If the representation ρ is fixed by α, then α will act, via its derivative $D(\alpha)_\rho$, on the tangent space $T_\rho(R_n F_d)$. In [3, Thm. 3.8, p. 62] it was shown that this derivative could be interpreted as the "Fox Jacobian" of α, namely the matrix $[\partial\alpha^{-1}x_i/\partial x_j]$. We recall this interpretation, and its connection with automorphism action on cocycles.

$R_n F_d$ can be identified with $GL_n\mathbf{C}^d$ via $\rho \mapsto (\rho(x_i))$. Since for any $A \in GL_n\mathbf{C}$ the tangent space $T_A(GL_n\mathbf{C})$ is $M_n\mathbf{C}$, $T_\rho(R_n F_d)$ can be identified with $M_n\mathbf{C}^{(d)}$, which we regard as column d-tuples of matrices. F_d acts on these d-tuples via $\mathrm{Ad} \circ \rho$ on each entry. We define the Fox Jacobian of $\alpha \in \mathrm{Aut}(F_d)$ as follows:

Definition 1. Let $\alpha \in \mathrm{Aut}(F_d)$. Then $(\partial\alpha^{-1}/\partial x)$ denotes the $d \times d$ $\left[\partial\alpha^{-1}(x_i)/\partial x_j\right]$; the entries here are Fox derivatives [1, §3.1] and hence elements of $\mathbf{Z}[F_d]$. We call this matrix the *Fox Jacobian* of α.

Given a representation ρ of F_d (and hence $\mathbf{Z}[F_d]$) we write $(\partial\alpha^{-1}/\partial x)^\rho$ for the action of $(\partial\alpha^{-1}/\partial x)$ on d-tuples from the representation space of ρ. In this notation, we recall the statement of [3, Thm. 3.8, p.62].

Jacobian Formula. *Let $\rho \in R_n F_d$ be fixed by $\alpha \in Aut(F_d)$. Then under the identification $T_\rho(R_n F_d) \to M_n\mathbf{C}^{(d)}$, $D(\alpha)_\rho = (\partial\alpha^{-1}/\partial x)^{\mathrm{Ad}\circ\rho}$.*

As pointed out in [3, Cor. 3.9, p.62], this identification is compatible with the identification [3, Prop. 2.2, p.33] of $T_\rho(R_n F_d)$ and $Z^1(F_d, \mathrm{Ad} \circ \rho)$, where Z^1 is identified with $M_n\mathbf{C}^{(d)}$ via $\sigma \mapsto [\sigma(x_i)]$. As we now note, cocycles in fixed representations (such as the above $\mathrm{Ad} \circ \rho$) admit an automorphism action:

Proposition 2. *Let M be a Γ-module and H a subgroup of $Aut(\Gamma)$. (Γ is any group). Assume that for all $x \in \Gamma$, $\alpha \in H$ and $m \in M$, we have $xm = \alpha(x)m$. Let $\sigma \in Z^1(\Gamma, M)$ be a cocycle. Then $\delta(\alpha)(\sigma) = \sigma \circ \alpha^{-1}$ is also a cocycle and $\delta : H \to GL(Z^1(\Gamma, M))$ is a group homomorphism.*

(We omit the elementary proof). As a corollary, we obtain for the free group F_d:

Corollary 3. *Let Γ in Proposition 2 be F_d. Then*

$$\delta(\alpha)(\sigma)(x_i) = \sum (\partial\alpha^{-1}(x_i)/\partial x_j)\sigma(x_j).$$

Moreover, if we identify $Z^1(F_d, M)$ with $M^{(d)}$ via $\sigma \mapsto [\sigma(x_i)]$ then $\delta(\alpha)$ on $M^{(d)}$ becomes multiplication by $(\partial\alpha^{-1}/\partial x)$.

Proof. $\delta(\alpha)(\sigma)(x_i) = \sigma(\alpha^{-1}x_i)$ which is expanded by the usual Fox formula [1, 3-2, p.105] to give the equation of the corollary. The matrix formula then follows.

For the free case considered in Corollary 3, we can be a little more explicit about the range of $\delta : H \mapsto GL(Z^1(F_d, M))$. As stated, we identify

$Z^1(F_d, M)$ and $M^{(d)}$, and the image of δ preserves this d-tuple structure. Thus in fact the image of δ lies in $GL_d(\mathrm{End}(M))$, the $d \times d$ matrices over the endomorphism ring of M. If $\rho_M : F_d \to GL(M)$ denotes the representation associated to M, and if R is a subring of $\mathrm{End}(M)$ containing $\rho_M(F_d)$, then the image of δ is in $GL_d(R)$. We will have occasion to use these observations below, where we refer to them as "restrictions on the image of δ associated to the restrictions on the image of ρ_M" (to lie in R).

Corollary 3 implies that at a fixed representation, the tangential action (Jacobian formula) is the same as the cocycle action (Proposition 2). To use either of these actions to construct representations δ of subgroups of $\mathrm{Aut}(F_d)$, we need fixed representations. We will actually be considering some subvarieties of fixed representations. We consider next the relevant formalities:

Let V be a closed subset of $R_n F_d$ with coordinate ring $A = \mathbf{C}[V]$. Associated to (the inclusion morphism into $R_n F_d$ from) V is a generic representation $P_V : F_d \to GL_n(A)$ defined by $\rho(x) = P_V(x)(\rho)$ for all $\rho \in V$, $x \in F_d$ [3, Prop. 1.19, p.19]. It follows from the defining formula that if every ρ in V is fixed by the subgroup H of $\mathrm{Aut}(F_d)$, then P_V is also fixed by H. Using Proposition 2, this then defines a representation δ of H in $GL(Z^1(F_d, \mathrm{Ad} \circ P_V))$, given by Corollary 3 by the matrix $(\partial\alpha^{-1}/\partial x)^{\mathrm{Ad} \circ P_V}$. For $\rho \in V$, $T_\rho(R_n F_d) = Z^1(F_d, \mathrm{Ad} \circ \rho) = M_n(\mathbf{C})^{(d)}$, and we can think of this latter as the evaluation of $M_n(A)^{(d)}$ at ρ. Since the action of F_d on $M_n(\mathbf{C})$ via $\mathrm{Ad} \circ \rho$ is given by evaluating the action of F_d on $M_n(A)$ via $\mathrm{Ad} \circ P_V$ at ρ, we have that the differential action of H on $T_\rho(R_n)$ is the evaluation of the action via δ on $Z^1(F_d, \mathrm{Ad} \circ P_V)$. We record this observation for later use.

Proposition 4. *Let V be a closed subset of $R_n F_d$ and let H be a subgroup of Aut(F_d). Assume that H fixes every element of V. Then the generic representation P_V of V is also fixed by H, and the representation $\delta : H \to GL(Z^1(F_d, Ad \circ P_V)$ satisfies $\delta(\alpha)(\rho) = (D\alpha)_\rho$ for all $\rho \in V$.*

Proposition 4 says that we can consider δ as a parameterized family of tangential representations (with parameter space V). If the H-invariant subset V of Proposition 4 is not only closed but also irreducible, its coordinate ring A, being an integral domain of the same cardinality as $\mathbf{C}$, admits a (non-$\mathbf{C}$-linear) embedding into $\mathbf{C}$, say $f : A \to \mathbf{C}$. Then we can consider the representation $\rho = GL_n(f)P_V : F_d \to GL_n\mathbf{C}$ obtained by applying f to the coordinates of P_V. This representation is also H-invariant (since P_V is) so we can consider the derivative action of H on $T_\rho(R_n F_d)$. As we now show, this is the same as the δ-representation of H on $Z^1(F_d, \mathrm{Ad} \circ P_V)$.

Proposition 5. *Let V be an irreducible closed subset of $R_n F_d$ and let H be a subgroup of Aut(F_d). Assume that H fixes every element of V. Let $f : \mathbf{C}[V] \to \mathbf{C}$ be an embedding and let $\rho = GL_n(f)P_V$. Then, for $\alpha \in H$, the map induced by f carries $\delta(\alpha)$ to $(D(\alpha))_\rho$.*

Proof. $\delta(\alpha)$ acts on $Z^1(F_d, \mathrm{Ad} \circ P_V) = M_n(A)^{(d)}$ via $(\partial\alpha^{-1}/\partial x)$, by

Corollary 3 (here $A = \mathbf{C}[V]$). $(D\alpha)_\rho$ acts on $T_\rho(R_nF_d) = M_n(\mathbf{C})^{(d)}$ by $(\partial\alpha^{-1}/\partial x)$ by the Jacobian formula. The action of F_d on $M_n(A)$ is by $\mathrm{Ad} \circ P_V$; the action on $M_n(\mathbf{C})$ is by $\mathrm{Ad} \circ \rho$. For $D \in M_n(A)$ and $x \in F_d$, we have $(\mathrm{Ad} \circ P_V)(x)(D) = P_V(x)DP_V(x)^{-1}$ so $(\mathrm{Ad} \circ \rho)(x)(M_n(f)(D)) = M_n(f)(\mathrm{Ad} \circ P_V)(x)(D))$. Thus the proposition follows.

Because of Proposition 5, the action of subgroups H of $\mathrm{Aut}(F_d)$ from irreducible subsets is just a special case of derivative actions in tangent spaces. In fact, since every closed subset is a union of finitely many irreducibles, every action from closed subsets is a finite product of derivative actions on tangent spaces.

To produce closed subsets of fixed representations, we can use the method of [1, §3.2]: we consider representations of quotients of F_d on which the automorphisms we study are trivial.

Proposition 6. *Let $\varphi : F_d \to G$ be a surjective homomorphism. Let H be a subgroup of $Aut(F_d)$ with $\varphi\alpha = \varphi$ for all $\alpha \in H$. Let V denote the image of R_nG in R_nF_d. Then V is a closed subset of R_nF_d fixed elementwise by H, the generic representation P_V of V factors through φ and the δ-representation of H on $Z^1(F_d, Ad \circ P_V)$ factors through the group homomorphism β_φ, from H to $GL_d(\mathbf{Z}[G])$ given by $\beta_\varphi(\alpha) = (\partial\alpha^{-1}/\partial x)^\varphi$.*

Proof. The map $\rho \mapsto \rho\varphi$ from R_nG to R_nF_d is a closed immersion [3, Prop. 1.7, p.8] and its image V is fixed by H. If P denotes the generic representation of R_nG, it is clear that $P_V = P\varphi$. Thus $\mathrm{Ad} \circ P_V$ factors through φ also, so that $\delta(\alpha) = (\partial\alpha^{-1}/\partial x)^{\mathrm{Ad}\circ P_V}$ acts by $(\partial\alpha^{-1}/\partial x)$ through φ, then P, and finally Ad.

The homomorphism β_φ of Proposition 6 can also be regarded as a special case of Corollary 3:

Proposition 7. *Let $\varphi : F_d \to G$ be a surjective homomorphism. Let H be a subgroup of $Aut(F_d)$ with $\varphi\alpha = \varphi$ for all $\alpha \in H$. Regard $\mathbf{Z}[G]$ as an F_d- module via φ. Under the identification of $Z^1(F_d, \mathbf{Z}[G])$ with $\mathbf{Z}[G]^{(d)}$, $\delta : H \to GL(Z^1(F_d, \mathbf{Z}[G])$ becomes β_φ.*

Proof. For $f \in \mathbf{Z}[G]$, $x \in F_d$ and $\alpha \in H$, we have $\alpha(x)f = \varphi(\alpha(x))f = \varphi(x)f = xf$, so Corollary 3 applies; it is clear that multiplication by $(\partial\alpha^{-1}/\partial x)$ on $\mathbf{Z}[G]^{(d)}$ acts via φ, hence by $(\partial\alpha^{-1}/\partial x)^\varphi = \beta_\varphi(\alpha)$.

Examples and Applications

The homomorphism β_φ of Proposition 6 is also used by Birman [1, Thm. 3.9, p.116] to explain the construction of some representations of subgroups of $\mathrm{Aut}(F_d)$. Proposition 5 shows that the parameterized family of tangential representations on the image of R_nG factors through (in a sense) the homomorphism β_φ. We examine Birman's examples from this point of view.

Example B. (The Burau representation of the braid group B_d). [2, p.536] B_d is the subgroup of all $\alpha \in \mathrm{Aut}(F_d)$ satisfying: α permutes and conjugates the generators $x_1, \ldots, x_d$, and preserves the product $x_1 \ldots x_d$ [1, Thm. 1.9, p.30]. Let $Z = \langle t \rangle$ be an infinite cyclic group and define $\varphi : F_d \to Z$ by $\varphi(x_i) = t$. The Burau representation is then $\beta_\varphi : B_d \to GL_d(\mathbf{Z}[t, t^{-1}])$ [1, Example 3, p.118]. As remarked in Proposition 7, β_φ here is the representation $\delta : B_d \to GL(Z^1(F_d, \mathbf{Z}[Z]))$. We want to consider this representation as a tangential one. We work with $n = 2$. The full image of R_2Z in R_2F_d will be too large for our purposes, and we consider instead the subset V of representations where t is represented by a diagonal elementary matrix:

$$V = \{\rho \in R_2F_d | \rho(x_i) = \mathrm{diag}(s, 1),\ s \in \mathbf{C}^*,\ 1 \le i \le d\}.$$

V has coordinate ring $\mathbf{C}[s, s^{-1}] = A$, is closed in R_2F_d, and is invariant under B_d. The generic representation P_V is given by $P_V(x_i)$ =diag$(s, 1))$ in $GL_2(A)$ so that $\mathrm{Ad} \circ P_V$ is given as follows on $M_2(A)$:

$$(\mathrm{Ad} \circ P_V)(x_i) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & sb \\ s^{-1}c & d \end{bmatrix}.$$

This equation shows that, as an F_d-module, $\mathrm{Ad} \circ P_V$ is isomorphic to a direct sum of four copies of A, with F_d trivial on the diagonal copies of A and acting via multiplication by s (or s^{-1}) on the off diagonal copies. But A, with F_d action by multiplication by s, is isomorphic to the group algebra $\mathbf{C}[Z]$ with F_d acting through φ. We thus have $\mathbf{C}[Z]$ as an F_d-module direct summand of $\mathrm{Ad} \circ P_V$. This makes $Z^1(F_d, \mathbf{C}[Z])$ a summand of $Z^1(F_d, \mathrm{Ad} \circ P_V)$. The representation of B_d on $Z^1(F_d\mathbf{C}[Z])$ is given by the Burau matrices $\beta_\varphi(\alpha) = (\partial\alpha^{-1}/\partial x)^\varphi$ by Proposition 7. The representation of B_d on $Z^1(F_d, \mathrm{Ad} \circ P_V)$ is given, by Proposition 4, as a parameterized family of derivative representations, with parameter $s \in \mathbf{C}^*$. By Proposition 5, this family is also realized at a single representation $\rho = GL_2(f)P_V$, where f is any embedding of A. Thus we conclude that the Burau representation is a direct summand of the tangential representation of ρ.

Example G. (The Gassner representation of the pure braid group P_d). [2, p.536] P_d is the subgroup of all $\alpha \in \mathrm{Aut}(F_d)$ satisfying: α conjugates the generators $x_1, \ldots, x_d$ and preserves the product $x_1 \ldots x_d$ [1, Cor. 1.8.3, p.25]. Let $Z_d = \langle t_1, \ldots, t_d \rangle$ be the free abelian group on $t_1, \ldots, t_d$ and define $\varphi : F_d \to Z_d$ by $\varphi(x_i) = t_i$. The Gassner representation is then $\beta_\varphi : P_d \to GL_n(\mathbf{Z}[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}])$ [1, Example 4, p.119]. This is realized tangentially as in Example B. First, we note that β_φ is also $\delta : P_d \to GL(Z^1(F_d, \mathbf{Z}[Z_d]))$. Then we consider the subset V of the image of R_2F_d in R_2F_d defined by $V = \{\rho \in R_2F_d | \rho(x_i) = \mathrm{diag}(s_i, 1),\ s_i \in \mathbf{C}^*,\ 1 \le i \le d\}$. V is closed in R_2F_d, invariant under P_d, and has coordinate ring $A = \mathbf{C}[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$. The generic representation P_V is given by $P_V(x_i) = \mathrm{diag}(s_i, 1)$ in $GL_2(A)$, so that $\mathrm{Ad} \circ P_V$ acts on $M_2(A)$ by

$$(\mathrm{Ad} \circ P_V)(x_i) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & s_i b \\ s_i^{-1} c & d \end{bmatrix}.$$

As before, since $A = \mathbf{C}[Z_d]$, we have $\mathbf{C}[Z_d]$ as an F_d-module direct summand of $\mathrm{Ad} \circ P_V$ so that $Z^1(F_d, \mathbf{C}[Z_d])$ is a direct factor of $Z^1(F_d, \mathrm{Ad} \circ \rho)$ so that the Gassner representation is a direct factor of the family of derivative representations parameterized by V, and hence a direct factor of a single tangential representation.

Now, we want to consider using fixed representations to represent "large" subgroups of $\mathrm{Aut}(F_d)$; that is, subgroups containing (at least) the inner automorphisms. We first remark that such representations are necessarily abelian:

Lemma 8. *Let $\rho \in R_n F_d$. Then ρ is fixed by all inner automorphism of F_d iff Kerρ contains (F_d, F_d).*

Proof. If $\mathrm{In}(y)$ fixes ρ, $\rho(y^{-1}xy) = \rho(x)$ for all $x \in F_d$, so $\rho(y^{-1}xyx^{-1}) = e$. If this holds for all y, ρ vanishes on all commutators. Conversely, if ρ is abelian, $\mathrm{Inn}(F_d)$ fixes ρ.

Next, we use a result of Blanchfield [1, Thm. 3.5, p.107] to determine when a δ representation vanishes on an inner automorphism.

Lemma 9. *Let M be an F_d-module, H a subgroup of Aut(F_d) containing Inn(F_d), and suppose H fixes the representation ρ afforded by M. Assume that 1 is not an eigenvalue of some $\rho(x_i)$. Then $\delta \circ In : F_d \to GL(Z^1(F_d, M))$ has kernel (Kerρ,Kerρ).*

Proof. We identify $Z^1(F_d, M)$ with $M^{(d)}$ so that, by Corollary 2, $\delta(\alpha)$ has matrix $(\partial\alpha^{-1}/\partial x)$ for $\alpha \in H$. If $\alpha = \mathrm{In}(y)$, then $\delta(\alpha)$ has matrix $\left(\partial(y^{-1}x_i y)/\partial x_j\right)^\rho$. Fox derivative calculations give that

$$\partial(y^{-1}x_i y)/\partial x_j = y^{-1}[(x_i - 1)\partial y/\partial x_j + \delta_{ij}]$$

so that the (i,j)-entry of $\delta(\mathrm{In}(y))$ is obtained by applying ρ to the above equation. By assumption, for some i $\rho(x_i) - 1$ is invertible. Then if $\delta(\mathrm{In}(y)) = I$, we have $\rho(\partial y/\partial x_j) = 0$ for all j. By Blanchfield's result [1, Thm. 3.9, p.10], we conclude that $y \in$ (Kerρ,Kerρ). Conversely, if $y \in$ (Kerρ,Kerρ), then $\rho(y^{-1}) = 1$ and, by [1, Thm. 3.7, p.107] again, $\rho(\partial y/\partial x_j) = 0$ for all j, so $\delta(\mathrm{In}(y)) = I$.

Lemma 9 allows us to determine the kernel of δ in these same circumstances.

Proposition 10. *Let M, H and ρ be as in Lemma 9. Then $\delta : H \to GL(Z^1(F_d, M))$ has the same kernel as the canonical map*

$$H \to \mathrm{A}ut(F_d/(Ker\rho, Ker\rho)).$$

Proof. If $\delta(\alpha) = I$ then the formula $\alpha \mathrm{In}(x)\alpha^{-1} = \mathrm{In}(\alpha(x))$ implies that $(\delta \circ I_n)(\alpha(x)) = (\delta \circ I_n)(x)$, so that $\alpha(x) \equiv x$ (modulo $(\mathrm{Ker}\rho)' =$ (Kerρ,Kerρ)).

Conversely, if $\alpha^{-1}(x_i) = x_i y_i$ where $y_i \in (\mathrm{Ker}\rho)'$ for $i = 1, \ldots, d$, then $(\partial\alpha^{-1}(x_i)/\partial x_j) = \delta_{ij} + (\partial(y_i)/\partial x_j)$ so that, applying ρ, we find using [1, Thm. 3.5., p.107] that $\delta(\alpha) = I$.

Because of Lemmas 8 and 9, the best we can hope for in a tangential representation of a subgroup of $\mathrm{Aut}(F_d)$ which contains $\mathrm{Inn}(F_d)$ is a kernel containing $\mathrm{Inn}(F_d'')$, where $F_d' = (F_d, F_d)$ and $F_d'' = (F_d', F_d')$. As we now show, we can achieve this by the methods of Example G.

Example I.I.A. Let $Z_d = \langle t_1, \ldots, t_d \rangle$ be the free abelian group on $t_1 \cdots t_d$, as in Example G. Let $s_1, \ldots, s_d$ in $\mathbf{C}$ be independent transcendentals so that $Z_d \to \mathbf{C}^*$ by $t_i \mapsto s_i$ is an injection. Define $\rho \in R_2 F_d$ by

$$\rho(x_i) = \begin{bmatrix} s_i & 0 \\ 0 & 1 \end{bmatrix}.$$

As in Example G, $(\mathrm{Ad} \circ \rho)(x_i)$ on $M_2\mathbf{C}$ preserves the subspace

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

and acts on it by multiplication by s_i. The resulting homomorphism $F_d \to GL_1\mathbf{C}$ factors through the above injection $Z_d \to \mathbf{C}^*$ so $\mathrm{Ker}(\mathrm{Ad} \circ \rho) = (F_d, F_d) = F_d'$. By Lemma 8, ρ is fixed by $\mathrm{Inn}(F_d)$. Of course, we also have $\mathrm{Ker}\rho = F_d'$. Thus $\alpha \in \mathrm{Aut}(F_d)$ fixes ρ if and only if $\alpha(x) \equiv x \pmod{F_d'}$ for all x, so that α is the identity modulo F_d'. It follows that the subgroup $K = \{\alpha \in \mathrm{Aut}(F_d) | \alpha = 1 \bmod F_d'\}$ is the stabilizer of ρ. By Proposition 10, the representation $\delta : K \to GL(T_\rho R_2 F_d)$ has the same kernel as $K \to \mathrm{Aut}(F_d/F_d'')$; we denote this kernel by H. The group K is called the group of $I - A$ automorphisms [1, Example 2, p.117] and K/H is the group of automorphisms of F_d/F_d'' induced by $I - A$ automorphisms of F_d [1, Example 2, p.117]. Our results now imply that:

The group of induced $I - A$ (I.I.A) automorphisms of F_d/F_d'' has a faithful linear representations in $GL(Z^1(F_d, Ad \circ \rho)) = GL_{4d}\mathbf{C}$.

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