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Strong Stabilization of Fourth-Order Plate Equations: A Semigroup Approach

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Abstract

Given Ω an open bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary Γ, we consider the nonhomogeneous Euler-Bernoulli equation in the solution $w(t, x)$:

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q = (0, \infty) \times \Omega \\ w(0, \ \cdot\ ) = w_0; w_t(0, \ \cdot\ ) = w_1 & \text{in } \Omega \\ w\big|_\Sigma = g_1 \in L^2(\Sigma) = L^2((0, \infty); L^2(\Gamma)) & \text{on } \Sigma = (0, \infty) \times \Gamma \\ \left.\dfrac{\partial w}{\partial v}\right|_\Sigma = g_2 = 0 & \text{on } \Sigma \end{cases} \tag{1}$$

We seek to express the non-zero control function g_1 as a suitable linear feedback applied to the velocity w_t, i.e., $w\big|_\Sigma = \mathbf{F}w_t$, such that $\mathbf{F}w_t \in L^2((0, \infty); L^2(\Gamma))$, and the corresponding closed loop system obtained by using such a feedback in (1) generates a (feedback) C_0-semigroup $e^{\mathbf{A}t}$ on $\mathbf{Z}$ which decays strongly to zero.

$$\|e^{\mathbf{A}t} z\|_{\mathbf{Z}} \to 0 \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad z \in \mathbf{Z}$$

Having identified the candidate $\mathbf{F}w_t = -\frac{\partial}{\partial v}[\Delta(A^{3/2} w_t)]$, where A is the operator defined by $Af = \Delta^2 f$; $\mathbb{D}(A) = \left\{ f \in L^2(\Omega) \colon \Delta^2 f \in L^2(\Omega), f\big|_\Gamma = \frac{\partial f}{\partial v}\big|_\Gamma = 0 \right\}$, we prove *strong stabilization*. Specifically, solutions of (1) go to zero in the strong topology of $\mathbf{Z}$: $\lim_{t\to\infty} \|[w(t), w_t(t)]\|_{\mathbf{Z}} = 0$, by the use of a Hilbert space decomposition for contractive semigroups.

1. Preliminaries, Choice of Stabilizing Feedback

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$ with sufficiently smooth boundary Γ. Consider the nonhomogeneous problem in the solution $w(t, x)$:

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q = (0, \infty) \times \Omega \quad \text{(a)} \\ w(0, \ \cdot\ ) = w_0 w_t(0, \ \cdot\ ) = w_1 & \text{in } \Omega \quad \text{(b)} \\ w\big|_{\Sigma} = g_1 \in L^2(\Sigma) = L^2((0, \infty); L^2(\Gamma)) & \text{on } \Sigma = (0, \infty) \times \Gamma \quad \text{(c)} \\ \dfrac{\partial w}{\partial v}\Big|_{\Sigma} = g_2 = 0 & \text{on } \Sigma \quad \text{(d)} \end{cases} \tag{1}$$

The goal of this paper is to obtain strong stabilization of the system (1) via a closed-loop feedback g_1 based on the velocity w_t. However, the optimal function space in which to work (a cross product space for position and velocity) is obtained from the exact controllability result to be summarized below.

First, we define the positive, self-adjoint opertor $A\colon \mathbb{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$Af \doteq \Delta^2 f \tag{2a}$$

$$\begin{aligned} \mathbb{D}(A) = \left\{ f \in L^2(\Omega) \colon \Delta^2 f \in L^2(\Omega), f\Big|_{\Gamma} = \frac{\partial f}{\partial v}\Big|_{\Gamma} = 0 \right\} \\ \equiv H^4(\Omega) \cap H_0^2(\Omega) \end{aligned} \tag{2b}$$

Since Ω is bounded in $\mathbb{R}^n$, then A has compact resolvent $R(\lambda; A)$. Also if $Af = 0$ for $f \in \mathbb{D}(A)$ then by Green's Theorem and (2a, b) we have

$$\begin{aligned} 0 = \langle Af, f\rangle_{\Omega} &= \langle \Delta(\Delta f), f\rangle_{\Omega} \\ &= \langle \Delta f, \Delta f\rangle_{\Omega} + \left\langle \frac{\partial(\Delta f)}{\partial v}, f \right\rangle_{\Gamma} - \left\langle \Delta f, \frac{\partial f}{\partial v} \right\rangle_{\Gamma} \\ &= \langle \Delta f, \Delta f\rangle_{\Omega} = \|\Delta f\|_{\Omega}^2 \end{aligned}$$

This implies $f = 0$; therefore,

$$A^{-1} \in \mathbf{L}(\Omega) \tag{3}$$

Next, we let

$$V = \left\{ f \in H^3(\Omega) \colon f\Big|_{\Gamma} = \frac{\partial f}{\partial v}\Big|_{\Gamma} = 0 \right\}. \tag{4a}$$

and consider the space $\mathbf{Z} = H^{-1}(\Omega) \times V'$. As shown in [L-T.2], $\mathbf{Z}$ can be characterized by using equivalent norms as

$$\mathbf{Z} = [\mathbb{D}(A^{1/4})]' \times [\mathbb{D}(A^{3/4})]' \tag{4b}$$

where $'$ denotes duality with respect to the $L^2(\Omega)$-topology.
The norms on these spaces ae given by

$$\|x\|_{\mathbb{D}(A^\alpha)} \doteq \|A^\alpha x\|_\Omega;\ \|x\|_{[\mathbb{D}(A^\beta)]'} \doteq \|A^{-\beta} x\|_\Omega \ \alpha, \beta \geq 0 \tag{5}$$

Below we state the regularity result as well as the exact controllability result.

Theorem (Regularity) [L.1], [L-T.2]**.** *Consider the problem* (1) *subject to* $[w_0, w_1] \in \mathbf{Z}$, $g_1 \in L^2((0,T); L^2(\Gamma))$, $g_2 \in L^2((0,T); H^{-1}(\Gamma))$. *Then the map* $\{w_0, w_1, g_1, g_2\} \to [w(t), w_t(t)] \in C([0,T]; \mathbf{Z})$ *is continuous for any* $0 < T < \infty$. □

Theorem (Exact Controllability) [L-T.1], [L-T.2]**.**

(i) *Assume there exists a point* $x_0 \in \mathbb{R}^n$ *such that* $(x - x_0) \cdot v \geq \gamma > 0$ *on* Γ *where* v *is the unit outward normal vector. Let* $0 < T < \infty$ *be arbitrary. If* $[w_0, w_1] \in \mathbf{Z}$ *arbitrary, then there exists a suitable control function* $g_1 \in L^2((0,T); L^2(\Gamma))$, *such that the corresponding solution of* (1) *with* $g_2 \equiv 0$ *satisfies* $w(T,\ \cdot\ ) = w_t(T,\ \cdot\ ) \equiv 0$ *and in addition* $[w, w_t] \in C([0,T]; \mathbf{Z})$.

(ii) *The same conclusion holds true without geometrical conditions if* g_2 *is taken within the class of* $L^2((0,T); H^{-1}(\Gamma))$ *controls.* □

By time reversibility, we see that at any finite T the totality of all solution points $\{w(T), w_t(T)\}$ of problem (1) with $w_0 = w_1 = 0$ fills all of the space $\mathbf{Z}$ when either g_1 runs over all of $L^2((0,T); L^2(\Gamma))$ and $g_2 \equiv 0$ under geometrical conditions on Ω, or else when $\{g_1, g_2\}$ runs over all of $L^2((0,T); L^2(\Gamma)) \times L^2((0,T); H^{-1}(\Gamma))$ without geometrical conditions. Therefore, since the space of exact controllability is the space of maximum regularity, we seek stabilization in exactly this space $\mathbf{Z}$.

We define the "energy" $E(t)$ for the dynamics (1) over the space $\mathbf{Z} = [\mathbb{D}(A^{1/4})]' \times [\mathbb{D}(A^{3/4})]'$ by

$$\begin{aligned} E(t) \doteq \left\| \left| \begin{matrix} w(t) \\ w_t(t) \end{matrix} \right| \right\|^2_{\mathbf{Z}} &\doteq \|w(t)\|^2_{[\mathbb{D}(A^{1/4})]'} \times \|w_t(t)\|^2_{[\mathbb{D}(A^{3/4})]'} \\ &\doteq \|A^{-1/4} w(t)\|^2_\Omega + \|A^{-3/4} w_t(t)\|^2_\Omega \end{aligned} \tag{6}$$

Next we seek a candidate g_1 which at least produces $\frac{dE}{dt} \leq 0$, i.e. energy "decrease." This does not, however, guarantee $\lim_{t\to\infty} E(t) = 0$, which is precisely strong stability of (1).

Remark 1. Below we shall show well-posedness in $\mathbf{Z}$, with $g_2 = 0$. Then since $w_t \in [\mathbb{D}(A^{3/4})]'$, it follows that $A^{-3/2} w_t = A^{-3/4} A^{-3/4} w_1 \in \mathbb{D}(A^{3/4}) \equiv V$. Therefore, $A^{-3/2} w_t$ satisfies the required boundary conditions

$$A^{-3/2} w_t \Big|_\Gamma = \frac{\partial}{\partial v} (A^{-3/2} w_t) \Big|_\Gamma \equiv 0 \tag{7}$$

By writing $E(t) = \langle A^{-1/4}w,\, A^{-1/4}w\rangle_\Omega + \langle A^{-3/4}w_t,\, A^{-3/4}w_t\rangle_\Omega$ and differentiating with respect to t, we have that

$$\frac{1}{2}\frac{dE}{dt} = \left\langle A^{-1/4}w,\, A^{-1/4}w_t\right\rangle_\Omega + \left\langle A^{-3/4}w_{tt},\, A^{-3/4}w_t\right\rangle_\Omega$$

by (1a)

$$= \left\langle w, A^{-1/2}w_t\right\rangle_\Omega - \left\langle \Delta(\Delta w), A^{-3/2}w_t\right\rangle_\Omega$$

by Green's Theorem

$$= \left\langle w, A^{-1/2}w_t\right\rangle_\Omega - \left[\left\langle \frac{\partial}{\partial v}(\Delta w), A^{-3/2}w_t\right\rangle_\Gamma - \left\langle \Delta w, \frac{\partial}{\partial v}(A^{-3/2}w_t)\right\rangle_\Gamma + \left\langle \Delta w, \Delta(A^{-3/2}w_t)\right\rangle_\Omega\right]$$

by (7)

$$= \left\langle w, A^{-1/2}w_t\right\rangle_\Omega - \left\langle \Delta w, \Delta(A^{-3/2}w_t)\right\rangle_\Omega$$

by Greens's Theorem

$$= \left\langle w, A^{-1/2}w_t\right\rangle_\Omega - \left[\left\langle \frac{\partial w}{\partial v}, \Delta(A^{-3/2}w_t)\right\rangle_\Gamma - \left\langle w, \frac{\partial}{\partial v}[\Delta(A^{-3/2}w_t)]\right\rangle_\Gamma + \left\langle w, \Delta^2(A^{-3/2}w_t)\right\rangle_\Omega\right]$$

$$= \left\langle w, A^{-1/2}w_t\right\rangle_\Omega + \left\langle w, \frac{\partial}{\partial v}[\Delta(A^{-3/2}w_t)]\right\rangle_\Gamma - \left\langle w, A^{-1/2}w_t\right\rangle_\Omega$$

$$= \left\langle w, \frac{\partial}{\partial v}[\Delta(A^{-3/2}w_t)]\right\rangle_\Gamma$$

Therefore, by selecting the simplest choice

$$w\Big|_\Sigma = g_1 = -\frac{\partial}{\partial v}[\Delta(A^{-3/2}w_t)], \tag{8}$$

we obtain $\frac{dE}{dt} = -2\|g_1\|_\Gamma^2 \le 0$, our desired energy decrease. □

Next we will show how our feedback can be expressed in terms of an operator (Green map) which acts from boundary Γ to interior Ω. Following [L-T.1], [L-T.2], we define $G_1\colon L^2(\Gamma) \to L^2(\Omega)$ by

$$G_1 g = y \text{ if and only if } \begin{cases} Ay = 0 & \text{in } \Omega & \text{(a)} \\ y|_\Gamma = g & \text{on } \Gamma & \text{(b)} \\ \left.\dfrac{\partial y}{\partial v}\right|_\Gamma = 0 & \text{on } \Gamma & \text{(c)} \end{cases} \tag{9}$$

We quote the following Lemma which will be used below.

Lemma [L-T.2]**.** *Let* $G_1^*\colon L^2(\Omega) \to L^2(\Gamma)$ *denote the continuous operator defined by* $\langle G_1 g, v\rangle_\Omega = \langle g, G_1^* v\rangle_\Gamma$, $g \in L^2(\Gamma)$, $v \in L^2(\Omega)$, *i.e.* G_1^* *is the adjoint of* G_1*. Then*

$$G_1^* A f = \frac{\partial}{\partial v}(\Delta f)\Big|_\Gamma \quad \textit{for} \quad f \in \mathbb{D}(A) \tag{10}$$

Now using (8) and (10) we see that

$$w\Big|_\Sigma = g_1 = -\frac{\partial}{\partial v}[\Delta(A^{-3/2} w_t)] = -G_1^* A(A^{-3/2} w_t) = -G_1^* A^{-1/2} w_t \tag{11}$$

Using elliptic theory [L-M, Vol. I, p. 188] we have that for any s real

$$G_1\colon \text{ continuous } H^s(\Gamma) \to H^{s+1/2}(\Omega) \tag{12a}$$

and in particular for $s = 0$

$$G_1\colon \text{ continuous } L^2(\Gamma) \to H^{1/2}(\Omega) \tag{12b}$$

We also have that by duality on (12a) with $s = -3/2$

$$G_1^*\colon \text{ continuous } H_0^1(\Omega) \to H^{3/2}(\Gamma) \tag{13}$$

so that (12a), (13) imply

$$G_1 G_1^*\colon \text{ continuous } \mathbb{D}(A^{1/4}) = H_0^1(\Omega) \to H^2(\Omega) \tag{14}$$

to be used below (17b) in the description of the domain of the feedback generator.

§2 Well-Posedness and Semigroup Generation

First we want to introduce an abstract operator model for problem (1). According to [T.1], [T.2], [L-T.1], [L-T.2], problem (1) with $g_2 = 0$ admits the following abstract versions:
as a second order equation

$$\ddot{w} = -A[w - G_1 g_1] = -A[w + G_1 G_1^* A^{-1/2} \dot{w}] \tag{15a}$$

or else as a first order system

$$\frac{d}{dt}\begin{vmatrix} w \\ \dot{w} \end{vmatrix} = \mathbb{A}\begin{vmatrix} w \\ \dot{w} \end{vmatrix};\ [w, \dot{w}] \in \mathbf{Z} = [\mathbb{D}(A^{1/4})]' \times [\mathbb{D}(A^{3/4})]' \tag{15b}$$

$$\text{where} \quad \mathbb{A} = \begin{vmatrix} 0 & I \\ -A & -AG_1 G_1^* A^{-1/2} \end{vmatrix} \tag{16}$$

More explicitly, if $y \in \mathbb{D}(\mathbb{A})$, then we can write

$$\mathbb{A}y = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} y_1 + G_1 G_1^* A^{-1/2} y_2 \\ y_2 \end{vmatrix} \tag{17a}$$

Thus, $\mathbb{D}(\mathbb{A}) = \big\{[y_1, y_2] \in \mathbf{Z}; y_2 \in [\mathbb{D}(A^{1/4})]'$ and $-A[y_1 + G_1 G_1^* A^{-1/2} y_2] \in [\mathbb{D}(A^{3/4})]'$, i.e. $y_1 + G_1 G_1^* A^{-1/2} y_2 \in \mathbb{D}(A^{1/4}) \equiv H_0^1(\Omega)$ which implies $y_1 \in H^1(\Omega)\big\}$. The operator $\mathbb{A}$ defined above is our candidate to be the generator of a feedback semigroup. The first step in this direction is the following Lemma.

Lemma 2. *The operator $\mathbb{A}$ is dissipative on $\mathbf{Z} = [\mathbb{D}(A^{1/4})]' \times [\mathbb{D}(A^{3/4})]'$.*

Proof. Let $z \in \mathbf{Z}$, then using below the skew-adjointness of $\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$, we have for $z \in \mathbb{D}(\mathbb{A})$

$$\begin{aligned} \operatorname{Re} \langle \mathbb{A}z, z \rangle_{\mathbf{Z}} &= \operatorname{Re} \left\langle \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\rangle_{\mathbf{Z}} \\ &+ \operatorname{Re} \left\langle \begin{vmatrix} 0 & 0 \\ 0 & -AG_1 G_1^* A^{-1/2} \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \right\rangle_{\mathbf{Z}} \\ &= 0 - \left\langle AG_1 G_1^* A^{-1/2} z_2, z_2 \right\rangle_{[\mathbb{D}(A^{3/4})]'} \\ &\qquad = \left\langle A^{-3/4} AG_1 G_1^* A^{-1/2} z_2, A^{-3/4} z_2 \right\rangle_{\Omega} \\ &= -\|G_1^* A^{-1/2} z_2\|_\Gamma^2 \le 0 \text{ and dissipativity holds.} \qquad \square \end{aligned}$$

The above proof is a reformulation of our argument below (7). Now we come to our result on semigroup generation.

Theorem 3.

(i) *The dissipative operator $\mathbb{A}$ in (16) also satisfies range $(\lambda I - \mathbb{A}) \equiv \mathbf{Z}$ on $\mathbf{Z}$ for $\lambda > 0$. Thus, by the Lumer-Phillips Theorem [P], $\mathbb{A}$ generates a C_0-semigroup of contractions $e^{\mathbf{A}t}$ on $\mathbf{Z}$, $t \ge 0$ and the solution of* (1), (11) *is given by*

$$\begin{vmatrix} w(t, w_0, w_1) \\ w_t(t, w_0, w_1) \end{vmatrix} = e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \quad \text{all } t \ge 0, \quad [w_0, w_1] \in \mathbf{Z} \tag{18a}$$

and in fact

$$\left\| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_{\mathbf{Z}}^2 = E(t) = \int_\Omega \left\{ |A^{-1/4} w|^2 + |A^{-3/4} w_t|^2 \right\} d\Omega \tag{18b}$$

(ii) *The resolvent operator $R(\lambda; \mathbb{A})$ of $\mathbb{A}$ is given by*

$$R(\lambda; \mathbb{A}) = \begin{vmatrix} \frac{I - V(\lambda)^{-1}}{\lambda} & V(\lambda)^{-1} A^{-1} \\ -V(\lambda)^{-1} & \lambda V(\lambda)^{-1} A^{-1} \end{vmatrix} \tag{19a}$$

$$\text{where} \quad V(\lambda) = [I + \lambda G_1 G_1^* A^{-1/2} + \lambda^2 A^{-1}] \tag{19b}$$

and $R(\lambda;\mathbb{A})$ is well-defined and compact on $\mathbf{Z}$ for all λ satisfying $Re\,\lambda \geq 0$. Thus, the spectrum (point) of $\mathbb{A}$ satisfies

$$\sigma(\mathbb{A}) \subset \{\lambda \colon Re\,\lambda < 0\} \tag{20}$$

Proof of Theorem 3. Dissipativity of $\mathbb{A}$ on $\mathbf{Z}$ was already shown in Lemma 2. Next, fix $\lambda > 0$ and let $z \in \mathbf{Z}$. We solve $(\lambda I - \mathbf{Z})y = z$, i.e.

$$\lambda y_1 - y_2 = z_1 \in [\mathbb{D}(A^{1/4})]' \tag{21a}$$

$$A(y_1 + G_1 G_1^* A^{-1/2} y_2) + \lambda y_2 = z_2 \in [\mathbb{D}(A^{3/4})]' \tag{21b}$$

for $y \in \mathbb{D}(\mathbb{A})$. We apply A^{-1} to (21b), multiply (21a) by λ and subtract to obtain

$$V(\lambda) y_2 = \lambda A^{-1} z_2 - z_1 \in [\mathbb{D}(A^{1/4})]' \tag{22}$$

with $V(\lambda)$ defined in (19b).

We next note that $V(\lambda)$ is boundedly invertible on $[\mathbb{D}(A^{1/4})]'$ since equivalently $A^{-1/4}V(\lambda)A^{1/4} = I + \lambda A^{-1/4} G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1}$ is boundedly invertible on $L^2(\Omega)$ (being self-adjoint, strictly positive on $L^2(\Omega)$) with inverse

$$A^{-1/4} V^{-1}(\lambda) A^{1/4} \in \mathbf{L}(\Omega) \tag{23}$$

Thus from (22)

$$y_2 = V^{-1}(\lambda)(\lambda A^{-1} z_2 - z_1) \in [\mathbb{D}(A^{1/4})]' \tag{24}$$

which then, inserted in (21a), yields

$$y_1 = \left[\frac{I - V^{-1}(\lambda)}{\lambda}\right] z_1 + V^{-1}(\lambda) A^{-1} z_2 \tag{25}$$

Then (19a) follows from (24) and (25). Note that from (21b) and (24)

$$y_1 + G_1 G_1^* A^{-1/2} y_2 = A^{-1} z_2 - \lambda A^{-1} y_2 \in \mathbb{D}(A^{1/4}) \tag{26}$$

So that, recalling (17b), we see that from (24) and (26) it is verified that $y \in \mathbb{D}(\mathbb{A})$. The compactness of $R(\lambda;\mathbb{A})$ on $\mathbf{Z}$ is readily seen from (19a) to be equivalent to compactness on $L^2(\Omega)$ of the following operators:

$$A^{-1/4}(I - V^{-1}(\lambda))A^{1/4} \tag{27a}$$

$$A^{-1/4}V^{-1}(\lambda)A^{-1/4} = A^{-1/4}V^{-1}(\lambda)A^{1/4}A^{-1/2} \tag{27b}$$

$$A^{-3/4}V^{-1}(\lambda)A^{1/4} = A^{-1/2}A^{-1/4}V^{-1}(\lambda)A^{1/4} \tag{27c}$$

$$A^{-3/4}V^{-1}(\lambda)A^{-1/4} = A^{-1/2}A^{-1/4}V^{-1}(\lambda)A^{1/4}A^{-1/2} \tag{27d}$$

First, compactness of the operators (27b-c-d) on $L^2(\Omega)$ is plain from (23) and $A^{-\alpha}$, $\alpha > 0$ being compact on $L^2(\Omega)$. For (27a) apply $V^{-1}(\lambda)$ on (19b) so that

$$I = V^{-1}(\lambda) + \lambda V^{-1}(\lambda) G_1 G_1^* A^{-1/2} + \lambda^2 V^{-1}(\lambda) A^{-1}$$

and then

$$\begin{aligned} &A^{-1/4}[I - V^{-1}(\lambda)]A^{1/4} \\ &\qquad = \lambda A^{-1/4} V^{-1}(\lambda) G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1/4} V^{-1}(\lambda) A^{-3/4} \\ &\quad = \lambda A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1/4} G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1} \end{aligned}$$

which is compact on $L^2(\Omega)$ by (23), since $A^{-1/4} G_1 G_1^* A^{-1/4} \in \mathbf{L}(\Omega)$. □

To complete the proof of Theorem 3, we must show that $\sigma(\mathbb{A})$ does not contain any points on the imaginary axis (we already know that $\sigma(\mathbb{A})$ does not contain points in $\{\operatorname{Re}\lambda > 0\}$ since $\mathbb{A}$ is the generator of a contraction semigroup).
Thus, we need to show that

$$V(\lambda)^{-1} \in \mathbf{L}([\mathbb{D}(A^{1/4})]') \quad \text{for} \quad \lambda = ir,\ r \in \mathbb{R},\ r \neq 0 \tag{28}$$

To this end let $x \in [\mathbb{D}(A^{1/4})]'$ and suppose $V(\lambda)x = 0$ for $\lambda = ir$. Then from (19b),

$$\begin{aligned} 0 &= \langle V(\lambda)x, x\rangle_{[\mathbb{D}(A^{1/4})]'} \\ &= \langle x, x\rangle_{[\mathbb{D}(A^{1/4})]'} + ir \left\langle G_1 G_1^* A^{-1/2} x, x \right\rangle_{[\mathbb{D}(A^{1/4})]'} - r^2 \left\langle A^{-1} x, x \right\rangle_{[\mathbb{D}(A^{1/4})]'} \\ &= \left\langle A^{-1/2} x, x \right\rangle_{\Omega} + ir \|G_1^* A^{-1/2} x\|_{\Gamma}^2 - r^2 \left\langle A^{-3/2} x, x \right\rangle \Omega \end{aligned} \tag{29}$$

Since the middle term in (29) is purely imaginary, we must have that, via (10),

$$G_1^* A^{-1/2} x\Big|_{\Gamma} = G_1^* A[A^{-3/2} x]\Big|_{\Gamma} = \frac{\partial}{\partial v}[\Delta(A^{-3/2} x)]_{\Gamma} = 0 \tag{30}$$

Also, we have that, by (29), $A^{-1/2}x = r^2 A^{-3/2} x$, i.e.

$$Ax = r^2 x \tag{31}$$

which means that x must be an eigenvector of A, say $x = e_n$ with eigenvalue r^2. Therefore, since $e_n \in \mathbb{D}(A)$, we have that it satisfies the two zero boundary conditions associated with $\mathbb{D}(A)$ (see (2b)), as well as (30). Therefore, the following Lemma will complete the proof of Theorem 3.

Lemma 4. *Let* $\lambda = r^2 > 0$*. The problem*

$$\begin{cases} \Delta^2 \phi = \lambda \phi & \text{in } \Omega \quad \text{(a)} \\ \phi\big|_\Gamma = 0 & \text{on } \Gamma \quad \text{(b)} \\ \dfrac{\partial \phi}{\partial v}\Big|_\Gamma = 0 & \text{on } \Gamma \quad \text{(c)} \\ \dfrac{\partial(\Delta \phi)}{\partial v}\Big|_\Gamma = 0 & \text{on } \Gamma \quad \text{(d)} \end{cases} \tag{32}$$

has only the trivial solution $\phi \equiv 0$*.*

Notes.

1. Since $A^{-3/2}x = A^{-3/2}e_n = r^{-3/2}e_n$, (32d) follows from (30).
2. The above lemma is not covered by standard elliptic theory, since only three boundary conditions, instead of four, are involved for the fourth-order elliptic operator in (32).

Proof of Lemma 4. Define the nonnegative self-adjoint operator B by

$$Bf \doteq \Delta^2 f;\ \mathbb{D}(B) = \left\{ f \in H^4(\Omega)\colon \frac{\partial f}{\partial v}\Big|_\Gamma = \frac{\partial(\Delta f)}{\partial v}\Big|_\Gamma = 0 \right\} \tag{33}$$

whose positive square root is defined by

$$B^{1/2} f \doteq -\Delta f;\ \mathbb{D}(B^{1/2}) = \left\{ f \in H^2(\Omega)\colon \frac{\partial f}{\partial v}\Big|_\Gamma = 0 \right\} \tag{34}$$

Now problem (32) can be rewritten as

$$\begin{cases} B\phi = r^2 \phi & \text{in } \Omega \quad \text{(a)} \\ \phi\Big|_\Gamma = 0 & \text{on } \Gamma \quad \text{(b)} \end{cases} \tag{35}$$

Thus either $\phi \equiv 0$ or ϕ is an eigenvector of B with eigenvalue r^2. Next multiply both sides of (35a) by $B^{-1/2}$ to obtain

$$B^{1/2}\phi = B^{-1/2} r^2 \phi = r^2 B^{-1/2} \phi = r^2 \left(\frac{\phi}{r} \right) = r\phi, \quad r > 0$$

So, recalling (34) and using (35b), we get the equivalent system

$$\begin{cases} -\Delta \phi = r\phi & \text{in } \Omega \\ \phi\Big|_\Gamma = 0 & \text{on } \Gamma \\ \dfrac{\partial \phi}{\partial v}\Big|_\Gamma = 0 & \text{on } \Gamma \end{cases} \tag{36}$$

Therefore, since (36) has only the trivial solution $\phi \equiv 0$, it follows that Lemma 4 and hence Theorem 3 are proved. $\square$

Now that we have proven that $\mathbb{A}$ generates a C_0-semigroup of contractions $e^{\mathbf{A}t}$ on $\mathbf{Z}$, it follows that

$$E(t) \le E(0) \quad \text{for} \quad t \ge 0 \tag{37}$$

This fact will be used crucially below. The next corollary is a consequence of the dissipative feedback perturbation on the boundary.

Corollary 5. *By choosing* $w\big|_\Sigma = g_1 = -G_1^* A^{-1/2} w_t$, *it follows that* $G_1^* A^{-1/2} w_t \in L^2((0,\infty); L^2(\Gamma))$ *and in fact*

$$\|G_1^* A^{-1/2} w_t\|^2_{L^2(\Sigma)} = \int_0^\infty \|G_1^* A^{-1/2} w_t\|^2_\Gamma dt \le \frac{1}{2} E(0) \tag{38}$$

for all initial conditions $[w_0, w_1] \in \mathbf{Z}$.

Proof of Corollary 5. Let $[w_0, w_1] \in \mathbb{D}(\mathbb{A})$ and recall for convenience

$$E(t) = \left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|^2_{\mathbf{Z}} = \left\| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|^2_{\mathbf{Z}} \quad \text{for} \quad t \ge 0 \tag{39}$$

Now

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E(t) &= \frac{1}{2}\frac{d}{dt}\left\langle e^{\mathbf{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}, \; e^{\mathbf{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\rangle_{\mathbf{Z}} \\ &= \mathrm{Re}\left\langle \mathbb{A}e^{\mathbf{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}, \; e^{\mathbf{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix}\right\rangle_{\mathbf{Z}} \\ &= \mathrm{Re}\left\langle \mathbb{A}\begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix}, \; \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix}\right\rangle_{\mathbf{Z}} \end{aligned}$$

via the proof of Lemma 2

$$= -\|G_1^* A^{-1/2} w_t\|^2_\Gamma \le 0 \tag{40}$$

Remark 6. We see that (40) shows that such a choice of g_1 does lead to an energy decrease as was demonstrated in another way (using Green's formula) in Remark 1. $\square$

Continuing the proof, now we integrate $\int_0^\infty dt$ both sides to obtain:

$$\begin{aligned} \int_0^\infty \|G_1^* A^{-1/2} w_t\|^2_\Gamma dt &= -\frac{1}{2} \lim_{T\to\infty} \int_0^T \frac{d}{dt}E(t)dt = \frac{1}{2} \lim_{T\to\infty} \int_T^0 \frac{d}{dt}E(t)dt \\ &= \frac{1}{2}E(0) - \frac{1}{2} \lim_{T\to\infty} E(T) \le E(0) \end{aligned}$$

where in the last equality we used the contraction of the semigroup, i.e. (37). Extension by continuity yields (38) for all $[w_0, w_1] \in \mathbf{Z}$. □

Theorem 7 (Strong Stabilization)**.** *For any* $[w_0, w_1] \in \mathbf{Z}$ *we have that*

$$E(t) = \left\| \begin{vmatrix} w(t, w_0, w_1) \\ w_t(t, w_0, w_1) \end{vmatrix} \right\|^2_{\mathbf{Z}} = \left\| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|^2_{\mathbf{Z}} \to 0 \text{ as } t \to +\infty \tag{41}$$

Proof of Theorem 7. The above result follows by appealing to the Nagy-Foias-Fogel decomposition theory [L.2]. Since $e^{\mathbf{A}t}$ is a C_0-contraction semigroup by Theorem 3, the Hilbert space $\mathbf{Z}$ can be decomposed in a unique way into the orthogonal sum:

$$\mathbf{Z} = \mathbf{Z}_{cnu} \oplus \mathbf{Z}_u \tag{42}$$

where both $\mathbf{Z}_{cnu}$ and $\mathbf{Z}_u$ are reducing subspaces for $e^{\mathbf{A}t}$ and its adjoint. It is also true that

(i) on $\mathbf{Z}_{cnu}$, $e^{\mathbf{A}t}$ is completely nonunitary and weakly stable,

(ii) on $\mathbf{Z}_u$, $e^{\mathbf{A}t}$ is a C_0-unitary group.

In our case, $\mathbf{Z}_u = \{0\}$, the trivial subspace, because otherwise an application of Stone's theorem [P] would guarantee at least one eigenvalue of $\mathbb{A}$ on the imaginary axis, but this is clearly false due to Theorem 3. Hence, $\mathbf{Z} \equiv \mathbf{Z}_{cnu}$ and therefore $e^{\mathbf{A}t}$ is weakly stable on $\mathbf{Z}$. However, since $\mathbb{A}$ has compact resolvent, it follows that $e^{\mathbf{A}t}$ is stable in the strong topology of $\mathbf{Z}$ [B]. Therefore, $e^{\mathbf{A}t} \to 0$ as $t \to +\infty$ for all $z \in \mathbf{Z}$ and strong stability is verified. □

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