Ulam Quarterly — Volume 1, Number 2, 1992

Alexandre Grothendieck's EGA V. Part II: Study of a Variable Hyperplane Section

(Interpretation and Rendition of his 'prenotes')

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§4 "Sufficiently General" Sections

We return to the general situation of Section 1, S being an arbitrary prescheme. Also, we suppose that X is of finite presentation over S.

In general, let us note that if $P(Z,k)$ is a "constructible" property of an algebraic *prescheme* Z over a field k then the set of $\xi \in P^v$ such that we have $P(Y_\xi, k(\xi))$ is *constructible* as we see by noting that Y_ξ is the fiber over ξ of $Y \to P^v$ which is a morphism of finite presentation and applying par. 9.[1] We have an analogous remark for a property $P(Z,M,k)$ where Z and k are as above and M is a coherent module over Z; if G is in addition of finite presentation over X then the set of $\xi \in P^v$ such that we have $P(Y_\eta, G_\eta, k(\eta))$ is constructible. On the other hand, in the previous two [Tr] sections we have developed in various cases a criterion for the preceding set E to contain the generic point of P^v, S being the spectrum of field k; *being constructible*, it follows that E contains a non-empty open

[1] of EGA IV. [Tr]

set: this is the passage of a conclusion from generic hyperplane section to the analogous conclusion for "sufficiently general" hyperplane sections.

Let us note in addition that in the case $S = \mathrm{Spec}(k)$ we also have a converse: in order that the generic hyperplane section should have the property P it is necessary and sufficient that the Y_ξ for ξ in a suitable neighborhood of η should satisfy it and it suffices to require for ξ *closed* in P^v (which for ξ k algebraically closed *leads* or *reduces* to considering k-rational points, i.e. hyperplane sections of X itself (without a prior extension to the base field.)(extension prealable Fr)

This follows, indeed, from the constructibility of E and from the fact that P^v is a Jacobson scheme.

Let us give as an example some special cases where the previous considerations apply:

Proposition 4.1. *Let G be a module of finite presentation over X.*

Let P be one of the following properties for a module M over an algebraic scheme Z over a field K;

(i) coprof $(M) \leq n$.

(ii) M satisfies (S_k).

(iii) M is Cohen Macauley.

(iv) M is without embedded prime cycle components.

(v) M is separable over K.

With these notations if E denotes the set of $\xi \in P$ such thatG_ξ satisfies property P then we have: (a) E is a constructible subset of P^v. (b) Let us suppose that $S = Spec(k)$, k a field, and that F satisfies property P. Let us also suppose that in athe case (v) that k is of characteristic 0 or that $f\colon X \to P$ is unramified, then E contains an open and dense suset of P^v.

Proof. (a) follows from the fact that P is a constructible property which we have seen in Par. 9 of EGA IV. As to (b), it follows from the corresponding results of the previous two sections.

Regrets. To (b): suppose more generally that if Z is the set of points of X where F does not satisfy P, we have $f(Z)$ is finite, i.e. dim $\overline{f(Z)} \leq 0$.

Proposition 4.2. *Let P be one of the following properties (for an algebraic prescheme over a field K):*

(i) Z is smooth over K.
(ii) Z satisfies the geometric property (R_k) over K.
(iii) Z is separable over K.
(iv) Z is geometrically normal over K.
(v) Z is geometrically integral over K.
(vi) Z is geometrically irreducible over K.

Let E be the set of $\xi \in P^v$ such that Y_ξ satisfies P. Then: (a) E is a constructible subset of P^v. (b) Let us suppose $S = Spec\, k$, k a field and let *us suppose in the cases (i) to (v) that k is of characteristic zero and that* $f: X \to P$ *is unramified. Finally, suppose in the cases (v) and (vi) that* $\dim \overline{f(X)} \geq 2$. *Assume that X satisfies P then E contains a dense open subset of* P^v.

Proof. Proof is identical to that of 4.2. Let us remark that in the cases (i) to (v) it suffices to assume *only* (in (b)) that $f(Z)$ is finite where Z is the set of points of X where P fails.

Corollary 4.3. *Let us consider the property* (C_m) *"*$\bar{Z}$ *cannot be disconnected by a closed subset of dimension* $\leq m$ *(where* $\bar{Z}$ *is* $Zx_K\bar{K}$, $\bar{K}$ *the algebraic closure of K)."*

Let E be the set of $\xi \in P^v$ *such that* Y_ξ *over* $K(\xi)$ *satisfies* C_m. *Then: (a) E is constructible. (b) Let us suppose that* $S = \mathrm{Spec}\,\underline{k}$, *k a field, and that for every irreducible component* X_i *of X we have* $\dim \overline{f(X)} \geq 2$. *Then if X over k satisfies* C_{m+1} *then* E_ξ *contains a dense open subset U of P.*

The constructibility is done by AQT[2] This is a fact that one has forgotten in Par. 9 of EGA IV that perhaps it would still be possible to repair (or fix); the part (b) follows in principle from 3.3 except that 3.3 has been announced in an excessively special manner and consequently should be generalized (the proof given being otherwise essentially unchanged). Also there is an error in the statement of 4.4, which is not valid as such if f is quasi-finite; in the general case instead of considering the dimension of the closed subsets of X respectively of Y_ξ it is sufficient to consider the dimension of their images in P respectively in H_ξ. Redactor demerdetur. [Latin] [Tr. Translate]

§5 Theorems of Seidenberg Type

5.1. In the present section we give conditions under which the set E defined in 4.1 is open. We deal here with properties of P of local nature over X, respectively Y_ξ, such that we can define the set U of $y \in Y$ so that (if ξ denotes the image of y in P^v) Y_ξ satisfies P at the point y (respectively G_ξ satisfies condition P at y). We give first of all the criteria for U to be open by using paragraph 12.[3] As always we have $E = P^v - h(Y - U)$ [Tr] it follows that if U is open and X is proper over S (since h is proper and a fortiori closed) then E is also open.[4]

5.2. As we have seen in No. 1 Y is defined in $Xx_SP^v = X_{P^v}$ as the "divisor" of a section ϕ of $\mathcal{O}_X(1) \otimes_S O_P^v(1)$ which induces for every $\xi \in P^v$ a section $\phi\xi$ of $\mathcal{O}_X(1) \otimes_{k(s)} O_P^v(1)(\xi)$ (a sheaf by the way isomorphic non-canonically to $O_X(1) \otimes_{k(s)} k(\xi) = O_{X_{k(s)}}(1)$) such that Y_ξ is nothing else but the "divisor" of this section (N.B. the divisor of a section ϕ of

[2] What is AQT? Ask AG.
[3] Locate that reference, most likely EGA IV [Tr], Yes [Tr].
[4] Since Y is proper over X and P^v is separated over S. (Marginal remark [Tr]).

an invertible module L is defined as the closed subscheme defined by the image ideal of $\phi - 1 : L^{-1} \to O$). If F is a sheaf of modules over X then its inverse image over Y, i.e. the inverse image of $F \otimes_{O_S} O_{P^v} = F_{P^v}$ over the subscheme Y of X_{P^v}, is nothing else but the cokernel of the homomorphism $\phi - 1 \otimes id_{F_{P^v}} : F_{P^v}(-1,-1) \to F_{P^v}$ where the notation $(-1,-1)$ explains itself as Mike[5] says. Also G_ξ is the cokernel of analogous homorphism $F_{k(\xi)}(-1,-1) \to F_{k(\xi)}$ where ξ is a point of P (and also we have a corresponding interpretation if ξ, instead of being a point of P^v, denotes a point of P^v with values in an S' over S ...)

In general if L is an invertible module somewhere, ϕ a section defining the subprescheme $V(\phi)$, then for every module F the inverse image of F *in* $V(\phi)$ can be identified, by the usual abuse of language, to the cokernel of $id_F \otimes^6 : F \otimes L^{-1} \to F$.

We say that ϕ is F regular if the preceding homomorphism is injective. If we choose an isomorphism of F and $\mathcal{O}_X$, which is possible locally, such that ϕ is identified to a section of O_X, this terminology is compatible with the one that was already introduced elsewhere.

Proposition 5.3. *With the previous notations let U be the set of $x \in X_P$ with image ξ in P^v such that $\phi\xi$ is $F_{K(\xi)}$ regular at x. Then*

(a) If F is of finite presentation and flat realtive to S then U is open and G/U is flat relative to P^v.

(b) For every $s \in P^v$ if η denotes the generic point of P^v_S then U contains $X_{k(\eta)}$.

Proof.

(a) Since F_{P^v} is of finite presentation and flat relative to P^v the conclusion follows from 11.3[7] (and also from o_{III} ... in the case of locally noetherian S) (of EGA IV [Tr]).

(b) We may suppose $S = \operatorname{Spec} k$. The associated cycles *to* $F_{k(\eta)}$ are (because of Par. 3)[8] the inverse images of associated cycles Z_i to F. If $f(Z_i)$ is finite, then by 2.3 $Z_{ik(\eta)} \cap Y = \phi$ in the contrary case by 2.6; for example, we also have $Z_{ik(\eta)} \cap Y = Z_{ik(\eta)}$ (by reason of dimension; 2.3 which was already involved in 2.6 and is without a doubt a better reason) which proves that ϕ does not vanish over any of the $Z_{ik(\eta)}$ and therefore proves (b).

Corollary 5.4. *Let V be the set of $\xi \in P^v$ such that $\phi\xi$ is $F_{k(\xi)}$ regular. If F is of finite presentation then V is constructible and it contains the generic points of the fibers of P^v over S. On the other hand, if also X is proper over S and F is flat over S, then the set V is open.*

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[5] Mike Artin (I presume P.B.)
[6] ??
[7] Find that reference.
[8] illegible, ask AG or figure out – probablyϕ^{-1} [Tr].

Remark 5.5. Let $\xi \in P^v$ over $s \in S$ and let us suppose tht F_s should be without associated embedded cycles. Then we see immediately that $\xi \in V$ (notation of 5.4) which means also that every irreducible component of $\operatorname{supp} F_{k(\xi)}$ does not lie over H_ξ (and a little less evidently in this criterion we replace $k(\xi)$ by an arbitrary extension of $k(\xi)$.

Let us note tht the hypothesis (S_1) aboutF_s which we have just made is satisfied notably if we suppose F_s Cohen-Macauley (a fortiori if F is CM over S); also in this case G_s is CM (since locally it is deduced from $F_{k(s)}$ which is such by dividing by $a\Phi \cdot F_{k(s)}$ where ϕ is $F_{k(s)}$ regular). The same remarks anyway should (and will have to)be made locally above to characterize the points of U (in place of those of V).

Using now 12.1.1 and 12.1.4[9] we obtain:

Theorem 5.6. *Let us assume tht F is of finite presentation flat relative to S. Let P be one of the properties (i) to (viii) of 12.1.1 or (if we assume $F = O_X$) one of the properties (i) to (iv) of 12.1.4 of EGA IV [Tr]. Let U_P be the set of $x \in X_P$ such that if ξ denotes the image of x in P^v the propertyP shuld be satisfied by G_ξ (resp. Y_ξ) at the point x and such that $\phi\xi$ is $F_{k(\xi)}$ regular at x. Then U_P is open and G/U_P is flat relative to S.*

Indeed, by the very definition we have $U_P \subset U$ (notation of 5.3 (a)) and we apply Par. 12 to $U \to P^v$ and F_{P^v}/U.

Corollary 5.7. *Let us suppose that $F$is of finite presentation flat relative to S, and $\operatorname{supp} F$ proper over S (e.g. X proper over S). Let V_P be the set of $\xi \in P^v$ such that G_ξ (resp. Y) satisfies the property P and that is $F_{K(\xi)}$ regular. Under these conditions V_P is open (and it is also constructible in every case, i.e. without any assumption of flatness or of properness).*

It seems to me that from the point of view of presentation we cannot leave 5.6 as is with a simple reference to conditions enumerated in another volume, but it requires an explicit list (i), (ii),... of properties which we have in view. Also remark (in 5.1 perhaps) that the case $P =$ geometrically normal (with $S = \operatorname{Spec}(k)$ for sure)[10] is due to Seidenberg.

§6 Connectedness of an arbitrary hyperplane section

We shall here combine the already known criterion of geometric connectedness of the generic hyperplane section (3.3) with the connectedness theorem of Zariski in order to obtain a connectedness result for an arbitrary hyperplane section:

Proposition 6.1. *We suppose $S = Spec(k)$, k an algebraically closed field [X proper over k suppose]*[11] *that for every irreducible component X_i of X,*

[9] Ask AG about reference – probably EGA IV [Tr]. 12.1.4 does not check out [Tr].

[10] or to be sure [Tr].

[11] illegible.

$\overline{f(X_i)}$ should be of dimension ≥ 2, finally that X cannot be disconnected by a closed subset Z of X such that dim $\overline{f(Z)} \leq 0$. *Under such conditions for every $\xi \in P^v$, Y_ξ is geometrically connected.*

Proof. Since none of the $f(X_i)$ is finite we see that every irreducible component Y_i of Y dominates P^v; on the other hand, $Y \to P^v$ is proper (if Y is proper over k, being such over X which is proper over k). On the other hand, by (3.3), the generic fiber Y_η of $Y \to P^v$ is geometrically connected.

Finally, P^v is regular and à fortiori geometrically unibranch. It now suffices to apply 15.6.3[12] (which is variant of the Zariski connectedness theorem) to conclude that *all* the fibers of $Y \to P^v$ are geometrically connected. q.e.d.

Indeed, it is not difficult by a proof of analogous type to generalize 6.1 in the same sense as in 4.4. If you do not want to trouble yourself with this exercise, at least mention it as a remark. To say also that we do not discriminate in 6.1 with regard to hyperplane sections that have an excessive (extra) dimension. (From the planning point of view) it might be clearer to group together all athe connectedness questions (including 3.3 and 4.4) in the same No. (or section).

§7 Application to the Construction of Special Hyperplane Sections and Multisections of Specified Type

7.1. Let us notice that if $S = \mathrm{Spec}(k)$ where k is an infinite field then every non-empty open subset of P^v contains a k-rational point; therefore in the notations of 4.1 if E (defined in terms of a constructible property P) contains the generic point η, it contains a k-rational point and therefore there exists a hyperplane section of X (defined over k) having the property P. On the other hand, S being again arbitrary, it is immediate that for every $s \in S$ and for every point ξ of the fiber P^v_s rational over $k(s)$, there exists a section ξ of P^v on an open neighborhood U of s which passes through ξ_0. If now E is again defined as in 4.1 in terms of a constructible property$P$ and if we have (for example due to No. 5) the fct that E is open, then if $\xi_0 \in E$, then the section ξ is a section of E over U at least if we sufficiently shrink or diminish U. Therefore we may construct a hyperplane section Y_ξ of X over an open neighborhood U of s such that for every $t \in U$ its fiber $Y_{\xi(t)}$ at t satisfies the property P. If we do not have à priori ξ_0 but if $k(s)$ is infinite we may combine the two preceding remarks to obtain a hyperplane section over an open neighborhood of s having the preceding property. Finally, using No. 5, we have a criterion allowing us to assert that (X resp. F being assumed flat over S which allows us to apply loc. cit.) Y_ξ resp.G_ξ is also flat over S. We may therefore, replacing X byY_ξ, iterate the previous construction which allows, for example under certain

[12] EGA IV [Tr].

conditions, to construct closer and closer (by successive approximations ???)[13] a "multisection" of S' of X over an open neighborhood U of the given point s, such that $S' \to U$ should be finite, flat and with fibers satisfying the property P. If $k(s)$ is finite we may be forced or constrained to do an étale and surjective base change $S' \to U$ (U an open neighborhood of s) before being able to apply the preceding constructions; indeed under the conditions from the start of 6.1, if k is finite there does not necessarily exist a rational point over k in the open non-empty set U, but there certainly exists a closed point of U, thus a point with values in a finite extension k' (necessarily separable) of k; when $k = k(s)$, therefore we may, after making a suitable finite étale extension S' over a neighborhood U of s, corresponding to the residual extension k', i.e. such that $S^1_s \xrightarrow{\sim} \mathrm{Spec}(k')$, restrict ourselves to the favorable situation of the unique point $s' \in S'$ over s after a base change $S' \to S$. I must however, note or point out [un remords Fr] a regret to 4.2 and 4.3 which should have been announced in a slightly more general form [at least as a remark]:If we are given an integer m and if we denote by E the set of $\xi \in P^v$ such tht G_ξ, resp. Y_ξ satisfies P exceptovr a closed set of dimension $\leq m$ (i.e. athe set P-singular Z is of dimension $\leq m$). Then

a) E is a constructible subset of P^v and
b) in the case $S = \mathrm{Spec}(k)$, if F, respectively X, satisfies P except over a set of dimension $\leq m + 1$, then E contains a non-empty open set.

Proposition 7.2. *Let us assume that X is proper over S and that F is of finite presentation finite and flat over S. Let P be one of the properties (i) to (v) of (4.2) and let m be an integer. Let $S \in S$ and let us suppose that the set Z_s of points of X_s where F_s does not satisfy P is of dimension $\leq m + 1$. Then if also $k(s)$ is infinite there exists a neighborhood U of s in E and a section ξ of P^v over Uh aving the following properties: For every$x \in U$ the set of points of $Y_{\xi,s}$ where $G_{\xi(s)}$ does not satisfy P is of dimension $\leq m$ and $\phi_{\xi(s)}$ is $F_{\xi,s}$ regular. Under such conditions the module G_ξ over Y_ξ is flat relative to U. Finally, if $k(s)$ is not supposed infinite, we can again make the previous construction afater an étale extension of the type anticipated in 7.1.*

Propostion 7.3. *Essentially the same. There is no longer an F and assume that X is flat relative to S we refer to properties (i) to (v) of 4.3 in place of those of 4.2 ("but being careful to make the reservation.") $k(s)$ of characteristic zero or $f\colon X \to P$ is an immersion and in the case (v) that for every $s \in S$ [illegible] irreducible component Z of X_s we have* dim $f(Z) \geq 2$. *[Nota Bene: For (v) compare 12.2.1 (x) and (xi) (we can then [illegible] in the other case 4.3 or 12.2.1 (x)] (marginal remark largely illegible in preceding square brackets).*

[13] Translator's note: de proche en proche [Fr].

(Text crossed out)

Proposition 7.4. *Let $g: X \to S$ be a flat proper morphism, let $s \in S$, let us put $n = \dim X_s$ and let us suppose that the dimension of the set of points of X_s where X_s is not separable over $k(s)$ is $\leq n$. (for example X_s separable). Then there exists an open neighborhood U of s and an étale finite, surjective morphism S'- - - -≻U such that $X\, x_S\, S'$ admits a section over S'. If $k(s)$ is infinite we may take for S' a closed subscheme of X U.*[14]

Let us assume to start with that $k(s)$ is infinite. We proceed by induction on n, the case $n = 0$ being trivial. Indeed in that case there exists an open neighborhood U of s such that $X|U$ itself is étale, finite and surjective above U as we see by immediate cross references. If $n > 0$, we apply 7.3 for the "separable" property which allows us to replace X by a "hyperplane section" Y having the same properties up to this that n is replaced by $n - 1$. If $k(s)$ is not assumed infinite we begin by making an étale base change, it works. (It goes thorugh)

Remark 7.5. In particular if X is projective and separable over S it admits locally over S étale multisections. But we note that we can give examples with X proper and smooth (but non-projective) S, where the same conclusion fails. Of course, the projective assumption cannot be weakened in general to an assumption of quasi-projectiveness as we see, for example, by taking X étale non-finite over S ...[15]

[14] Unclear, ask AG.
[15] Illegible