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The Étale Cohomology of p-Torsion Sheaves, II*

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Dedicated to the late Floyd Perkins,
a committed mathematics educator and mathematical physicist

Abstract

A formula due to Grothendieck, Ogg, and Shafarevich gives the Euler-Poincaré characteristic of a constructible sheaf of F_ℓ-modules on a smooth, proper curve over an algebraically closed field k of characteristic $p > 0$, as a sum of a global term and local terms, where $\ell \neq p$. A previously known result removes the restriction on ℓ in the case of p-torsion sheaves trivialized by p-extensions. The author conjectured a p-torsion analogue of the formula in an earlier paper and proved it there for the case of a constructible sheaf of F_p-modules when the generic stalk has rank p.

A general proof of the conjecture is given in the presence of the Étale Core Hypothesis (ECH). The étale core of a suitable finite, flat, height 1, commutative, p-torsion group scheme is the subsheaf fixed by the p-th power endomorphism of the tangent space at the identity and it is an étale group scheme. Étale cohomology with coefficients in an étale core is relatively easy to compute and can be used to compute the cohomology with coefficients in a p-torsion constructible sheaf.

Results on surfaces and the F_q-vector schemes of Raynaud are included, where $q = p^r$, $r \geq 1$. In the appendix, we establish conditions under which an étale core can be found via a finite étale extension of the finite étale group scheme corresponding to the generic stalk of a p-torsion sheaf.

Introduction

The étale cohomology of a scheme of characteristic $p > 0$ with coefficients in a p-torsion constructible sheaf is poorly understood in contrast

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to the case of a sheaf with torsion prime to the characteristic. The latter theory has been extensively covered; see for example [7] and [8] or [15]. One result that has been proved [12, 1.1] is a generalization of the Grothendieck-Ogg-Shafarevich (G-O-S) formula [18]. (Other authors have proved variants using étale cohomology [1], Galois cohomology [14], and the Cartier operator [20].) The formula expresses the Euler-Poincaré characteristic of a constructible sheaf of F_ℓ-modules on a smooth, proper curve, over an algebraically closed field k of characteristic $p > 0$, as a sum of local terms and a global term, where $\ell \neq p$. The generalization removes the restriction on ℓ in the case of a sheaf trivialized by a p-extension.

Utilizing the concept of the étale core, the author gave complete results on the Euler-Poincaré characteristic of a constructible sheaf of F_p-modules on a smooth, proper curve over k, when the generic stalk has rank p. This made possible such calculations as $H^2(X_{et}, F_p)$ for certain smooth, complete elliptic surfaces X over k and the p-ranks of smooth, proper curves that can be obtained as cyclic or abelian extensions over a smooth, proper base curve.

The étale core of an appropriate group scheme serves as an approximation, with cohomology that is relatively easy to compute, of a given p-torsion constructible sheaf on a scheme of characteristic p. The question arises as to the applicability of this method beyond the case when the generic stalk has rank p on a quasi-compact integral scheme with an ample divisor [12, 2.5]. It was conjectured that the method applies to any p-torsion sheaf on an integral scheme. The proof of the conjecture for various cases appears in Section I and the Appendix (see below).

We begin Section I by recalling the theory of the étale core from [12, Section II]. The étale core of a finite, flat, height 1, commutative, p-torsion group scheme is the subsheaf fixed by the p-th power endomorphism of the tangent space at the identity and it is an étale group scheme. This is Proposition 1.1. The cohomology with coefficients in an étale core is computed via Corollary 1.2. The conjecture is stated in this section (1.3). The most useful case of the conjecture is proved in Proposition 1.4.

In Section II, the Étale Core Hypothesis (ECH) is stated, i.e. that Conjecture 1.3 holds for a given integral scheme X, and proved in certain cases (2.2). This is the main result of the entire paper. A general theorem (2.3) states the étale cohomology of any constructible sheaf of F_p-modules on such a scheme in terms of the corresponding étale core. The p-torsion analogue of the G-O-S formula is given in Corollary 2.4. A result on the cohomology of F_p over a complete smooth surface X is given in 2.5.

The next section (III) concerns the F_q-vector schemes of Raynaud and their cohomology over smooth, proper curves ($q = p^r$, $r \geq 1$). After discussing finite étale F_q-vector schemes over the generic point η and height 1 schemes of F_q-vectors over an integral scheme X, we show there is a flat extension which is also a scheme of F_q-vectors, when X is a quasi-compact integral scheme with an ample divisor (3.1). Assuming a flat extension exists, one can show it is a scheme of F_q-vectors when X is a regular, irreducible scheme of dimension 1 (3.2). The cohomology of the corresponding étale core is given in 3.3. The cohomology of any constructible sheaf whose generic stalk corresponds to a finite étale F_q-vector scheme over η is calculated in 3.4. The specialized case of $X = P^1$ is considered in 3.5. We close Section III with some examples of these calculations over P^1.

Conditions for the existence of an étale extension are considered in the Appendix. We first use the theory of the fundamental group of a scheme and the notion of purity to show that a finite étale extension to an integral scheme from an open subscheme U exists when the complement of U has codimension ≥ 2 and all the local rings at points of the complement are regular. The extension is also finite étale when the complement of U is a local set-theoretic complete intersection that has codimension ≥ 3. (For both of these results, see A.1.) We prove a relative result (A.2) using the concept of étale (resp. homotopical) Z-depth (Z = the complement of U).

It is natural to consider generalizations of the G-O-S formula to higher dimensions. One can view the G-O-S formula as a generalization to étale cohomology of the Hurwitz genus formula [9, IV.2.4]. One approach is thus to seek a higher-dimensional analogue of the Hurwitz formula in characteristic 0. On the other hand, there is a result due to Laumon [13] which generalizes the G-O-S formula itself to the case of a surface. It applies to sheaves of F_ℓ-modules, $\ell \neq p$. In particular, it must hold for sheaves trivialized by ℓ-extensions. Since his result is stated in terms of cohomology with compact supports for what is essentially an open dense subscheme of a smooth surface, another form for complete surfaces, but not involving the Swan conductor, is needed before applying it to p-torsion sheaves.

In fact, it is possible to conjecture the following statement extending the G-O-S formula to a complete smooth variety of dimension n.

Conjecture. Let ℓ be any prime, X a complete smooth variety of dimension n over an algebraically closed field k of characteristic $p > 0$, and F a constructible sheaf of F_ℓ-modules on X_{et}. For $i = 0, \ldots, n$, denote by X^{n-i} the set of **generic** points of X of dimension $n - i$. If η denotes the generic point of X, set $c_\eta(F) = dim_{F_\ell}(F_{\bar{\eta}})$. Then, for $\ell \neq p$,

$$\chi(X_{et}, F) = \sum_{i=0}^{n} (-1)^i \sum_{x \in X^{n-i}} \chi(\overline{\{x\}}, F_\ell)\, c_x(F).$$

If F is trivialized by an ℓ-extension, then the same formula holds for any prime ℓ with $c_x(F)$ replaced by $t_x(F)$ and $t_\eta(F) = c_\eta(F)$. Moreover, when $\ell = p$ and X satisfies the ECH, let G be the corresponding étale core. Then

$$\chi(X_{et}, F) = \chi(X_{et}, G) - \sum_{i=1}^{n} (-1)^i \sum_{x \in X^{n-i}} \chi(\overline{\{x\}}, F_p)\, ((t_x(G) - t_x(F)).$$

The conjecture holds for $n = 0, 1$: see [15, V.2.12] for the case of $\ell \neq p$ and [12, 1.1] for the case of F trivialized by an ℓ-extension. The notation $t_x(F)$ is given before Corollary 2.4 below with p replaced by ℓ. For $n = 2$, it should give the result of Laumon.

For smooth, proper curves, only one more result is needed to complete the circle of ideas begun in [12, Section II]. The theory of the étale core (2.4) should be applied to give an alternative proof of the formula for p-torsion sheaves trivialized by p-extensions, i.e. Theorem 1.1 there. Then the known results on the étale cohomology of p-torsion sheaves would be bound by a common thread. The author hopes to have such a proof at some future time.

Notation and Conventions. All rings are commutative, Noetherian with 1. All schemes are locally Noetherian of characteristic $p > 0$ (unless otherwise specified). A variety is an integral, separated scheme of finite type over a field k. A curve is a variety of dimension 1; a surface is a variety of dimension 2. A proper variety over k is also called complete. **The field k will always be algebraically closed of characteristic $p > 0$.** If K is any field, then K_s will denote its separable closure. The finite field of ℓ elements is denoted F_ℓ, ℓ any prime. For an integral scheme X, we write $k(X)$ for the field of rational functions of X. The étale site X_{et} means the small étale site on the scheme X. The Cartier dual of a group scheme G is denoted G^D.

References will be enclosed in brackets [ ], except for internal cross-references. The symbol // will indicate the end of a proof.

Section I. the Étale Core Conjecture

We recall the concept of an étale core and how to compute its cohomology. A conjecture is stated about when an étale core can be found for a constructible sheaf of F_p-modules on an integral scheme. A useful form is proved. We establish notation at the end of this section that will be used again in the Appendix.

Let X be a scheme of characteristic $p > 0$ and A a group scheme over X. We assume all group schemes are commutative and killed by p. The tangent space at the identity will be denoted $\underline{Lie}(A)$. It has the structure of a p-Lie algebra [6, VIIA.6] with p-th power endomorphism $[p]$. The étale core of $\underline{Lie}(A)$ (or A) is the subsheaf fixed by $[p]$. If G is the étale core of A, then $G = \underline{Hom}_{X-grp}(A^D, F_p)$. The following proposition gives a characterization of the étale core.

Proposition 1.1. *Let A be a finite, flat, height 1 group scheme on X. Then $G = \underline{Hom}_{X-grp}(A^D, F_p)$ is an étale group scheme on X and there is a short exact sequence*

$$0 \to G \to \underline{Lie}(A) \xrightarrow{[p]-1} \underline{Lie}(A) \to 0$$

of sheaves on X_{et}.

This is Theorem 2.1 of [12]. The condition that A be height 1 is unnecessary since it is well-known that $\underline{Lie}(A) \cong \underline{Lie}({}_{FR}A)$, where ${}_{FR}A$ is the height 1 group scheme that is the kernel of the Frobenius FR: $A \to A^{(p)}$. If $G_x = G \times_X Spec\, k(x)$ for $x \in X$, then it is easy to see that G_x is finite étale when A_x^D is finite étale and $G_x = Spec\ k(x)$ otherwise. It is also true that every finite étale group scheme killed by p occurs as the étale core of a height 1 group scheme A, namely $A = (G^*)^D$. Here $G^* = \underline{Hom}_{X-grp}(G, F_p)$.

Remark. If A is a reduced, algebraic group scheme over a field k, then the order of the group of k-valued points of the étale core of A is an isogeny invariant, denoted $\sigma(A)$ in [17, II.8.6]. This follows from the isomorphisms $Hom(A^D, F_p) \cong Hom(\mu_p, A) \cong (Z/pZ)^{\sigma(A)}$. If A is an abelian variety over k, then this number is also the p-rank of A [17, II.12.12].

Now, the étale core sequence of Proposition 1.1 allows us to easily compute cohomology with coefficients in any étale core. We need some notation.

Notation. If V is a vector space of dimension d over a field k, let V_{ss} denote the semi-simple subspace of V under a p-linear, additive endomorphism f. Then $V_{ss} = \bigcap_{m\geq 1} Im(f^m)$ and $\bigcup_{m\geq 1} Ker(f^m)$ is denoted V_n.

The k-dimension of V_{ss} equals the F_p-dimension of V^f, the set of all $v \in V$ such that $f(v) = v$. If B is the matrix of f with respect to any basis, then the k-dimension of V_{ss} equals the rank of the matrix $B\,B^{(p)}\,\cdots\,B^{(p^{d-1})}$ by [11] and is sometimes called the stable rank of the matrix B.

Corollary 1.2. *Let X be a complete smooth variety over a field k. Then*

$$dim_{F_p}(H^i(X_{et}, G)) = dim_k H^i(X_{et}, \underline{Lie}(A))_{ss}$$

for all i.

This is Corollary 2.3 of [12]. As a special case, it provides a method of calculating $\chi(X_{et}, G) \overset{\text{def}}{=} \sum(-1)^i dim_{F_p} H^i(X_{et}, G)$ for any p-torsion, finite, étale group scheme G, when X is a complete smooth variety over k.

The following conjecture addresses the question of the applicability of these methods to computing the cohomology of p-torsion constructible sheaves.

Conjecture 1.3. Let X be an integral scheme and F a constructible sheaf of F_p-modules on X_{et} with $K = k(X)$ and $\eta = Spec\ K$. Then there exists a finite, flat, height 1 group scheme A on X whose étale core G has as generic fiber G_η the finite étale group scheme M corresponding to the generic stalk $F_{\bar{\eta}}$ of F.

It has been shown in [12] that the conjecture is true for any quasi-compact integral scheme with an ample divisor when the generic stalk of F has rank p. One knows that a constructible sheaf F on an integral scheme X determines a finite étale group scheme M over the generic point η [15, V.1.1, 1.8]. Hence, we need to understand how to extend finite étale K-group schemes M to X, where $K = k(X)$. There is an open subscheme U of X to which there is a finite étale extension by [15, V.1.8].

Proposition 1.4. *Let X be a regular quasi-projective scheme of characteristic $p > 0$ over a ring R with $K = k(X)$ and $\eta = Spec\ K$. If F is a constructible sheaf of F_p-modules on X_{et}, then the Cartier dual of the finite étale K-group scheme corresponding to the generic stalk $F_{\bar{\eta}}$ of F extends to a finite flat group scheme of height 1 over X.*

Proof. Let M denote the finite étale K-group scheme corresponding to the generic stalk of F. As M is p-torsion and étale unipotent, its Cartier dual M^D is infinitesimal multiplicative of height 1. By Proposition B.4 of [16], M^D extends to a finite flat group scheme A of height 1 over X. (Equivalently, M extends to A^D.)//

Additional situations for which 1.3 holds will be discussed in the Appendix.

We fix notation now for the Appendix. Let X be an integral scheme and H_1 a finite étale K-group scheme, where $K = k(X)$. Then H_1 extends to a finite étale group scheme H over some open subscheme U of X, i.e. $H_1 = H \times_U Spec\ K$.

Section II. The Étale Core Hypothesis

This is the main section of the paper, since the Étale Core Hypothesis (ECH) is proved here for certain cases. We begin with a discussion of finding an étale core that corresponds to a given constructible sheaf of F_p-modules, in the presence of the ECH. Then the various cases that have been proved are given. A general result on computing cohomology is stated and the p-torsion formula is established in a corollary. An application to surfaces is presented.

An integral scheme X will be said to satisfy the Étale Core Hypothesis (ECH) when Conjecture 1.3 holds for X. Let $K = k(X)$ and $\eta = Spec\ K$ be the generic point.

(ECH) For any constructible sheaf F of F_p-modules on X_{et}, there exists a finite, flat (height 1) commutative p-torsion group scheme A on X whose étale core G has as generic fiber G_η the finite étale K-group scheme M corresponding to $F_{\bar{\eta}}$.

Proposition 1.4 showed that under certain conditions we can get a finite flat extension to all of X. The existence of such an extension to all of X will be sufficient to imply the ECH. Thus, the proof of that fact in the following proposition clarifies the relationship between these ideas. (The results of the Appendix concern finite étale extensions from an open subscheme.)

Proposition 2.1 (Notation as above)**.** *If every finite étale K-group scheme M has a finite flat extension to an integral scheme X, then X satisfies the ECH.*

Proof. Denote by M^* the étale dual of a finite, étale, p-torsion, K-group scheme, i.e. $M^* = \underline{Hom}_{K-grp}(M, F_p)$. If M corresponds to $F_{\bar{\eta}}$, let A^D be a finite flat extension of M^* to X. Let $G = \underline{Hom}_{X-grp}(A^D, F_p)$ be the étale core of A as in Proposition 1.1. Then $G_\eta = G \times_X \eta \cong \underline{Hom}_{K-grp}(A^D_\eta, F_p) \cong \underline{Hom}_{K-grp}(M^*, F_p) = M^{**} \cong M$. This shows that G is an étale core whose generic fiber corresponds to $F_{\bar{\eta}}$. Thus, X satisfies the ECH.//

Corollary 2.2. *If the hypotheses of 1.4, A.1, or A.2 are satisfied, or if the generic stalk of F has rank p or type $(p, p, \ldots, p)$ and X is quasicompact with an ample divisor, then X satisfies the ECH.*

Proof. The first statement is clear and the second is [12, 2.5] or (3.1) below.//

This shows that the class of integral schemes satisfying the ECH is not empty. We now turn to the calculation of cohomology. If U is an open subscheme such that $G_{|U} \cong F_{|U}$ and F, G are locally constant on U, we denote by $j: U \to X$ and $i: Z = X - U \to X$ the corresponding open (respectively, closed) immersions.

Theorem 2.3. *Let X be a complete smooth scheme over a field k. Suppose X satisfies the ECH and F is a constructible sheaf of F_p-modules on X_{et}. If G is the corresponding étale core, then*

$$\chi(X_{et}, F) = \chi(X_{et}, G) - \chi(Z_{et}, i^*G) + \chi(Z_{et}, i^*F).$$

Proof. There is an exact sequence

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

of sheaves on X_{et}; similarly for G. Hence $\chi(F) = \chi(j_! j^* F) + \chi(i_* i^* F)$. Since we know $F_{\bar{\eta}} \cong G_{\bar{\eta}}$ $(= G_\eta(K_S))$ and $F_{|U} \cong G_{|U}$, we have $j^* F = F_{|U} \cong G_{|U} = j^* G$. Also, i_* is exact ([15, II.3.6]) so $H^s(X_{et}, i_* i^* F) \cong H^s(Z_{et}, i^* F)$; similarly for G. It follows that $\chi(X, F) - \chi(Z, i^* F) = \chi(X, j_! j^* F) = \chi(X, j_! j^* G) = \chi(X, G) - \chi(Z, i^* G)$. Hence, $\chi(X, F) = \chi(X, G) - \chi(Z, i^* G)$ $+ \chi(Z, i^* F)$.//

Definition. If F is a construcible sheaf of F_p-modules on X_{et} for a smooth curve X over a field k, then the **tame conductor** of F at $x \in X$ is $t_x(F) = dim(F_{\bar{\eta}}) - dim(F_{\bar{x}})$ $(dim = dim_{F_p})$.

Corollary 2.4 (p-Torsion Formula)**.** *If X is a complete smooth curve over k (as above), then*

$$\chi(X_{et}, F) = \chi(X_{et}, G) + \sum_{x \in X^0} t_x(G) - \sum_{x \in X^0} t_x(F).$$

Proof. Since Z is now a finite set, we have $H^s(Z_{et}, i^*F) = 0$ for $s > 0$ and $H^0(Z_{et}, i^*F) \cong H^0(\coprod_{x\in Z} x, i^*F) \cong \prod_{x\in Z} F_{\bar{x}}$; similarly for G. By the theorem, $\chi(X,F) = \chi(X,G) - \sum_{x\in X-U} dimG_{\bar{x}} + \sum_{x\in X-U} dimF_{\bar{x}}$. We know $F_{\bar{\eta}} \cong G_{\bar{\eta}}$ and so $\chi(X,F) = \chi(X,G) + \sum_{x\in X-U} (dimG_{\bar{\eta}} - dimG_{\bar{x}}) - \sum_{x\in X-U} (dimF_{\bar{\eta}} - dimF_{\bar{x}})$. The choice of U, i.e. F, G locally constant on U ensures that $\sum_{x\in X-U} dimF_{\bar{\eta}} - dimF_{\bar{x}} = \sum_{x\in X^0} dimF_{\bar{\eta}} - dimF_{\bar{x}} = \sum_{x\in X^0} t_x(F)$ with a similar result for G. Thus, $\chi(X,F) = \chi(X,G) + \sum_{x\in X^0} t_x(G) - \sum_{x\in X^0} t_x(F)$.//

In this p-torsion version of the G-O-S formula, the term $\chi(X,G)$ is a global term and the others are local terms.

We now look at an application of Corollary 2.4 to the study of surfaces. The argument is similar to that of [12, 3.3] and only a sketch of the proof will be given.

Theorem 2.5. *Let X be a complete smooth surface over a field k. Assume X has a proper flat morphism $\pi: X \to P^1$ whose generic fiber is a smooth curve of genus > 1 with non-zero p-rank. Let $K = k(P^1)$, $\Gamma_s = Gal(K_s/K)$, and $F = R^1\pi_*F_p$. Assume the action of Γ_s on the generic stalk of F factors through a finite quotient Γ whose maximal abelian quotient Γ_{ab} has order prime to p. Then*

$$\chi(X_{et}, F_p) = 1 - \chi(P^1, R^1\pi_*F_p)$$
$$= 1 - dim_k H^0(P^1, \bigoplus_{i=1}^{s} O(n_i))_{ss} + dim_k H^1(P^1, \bigoplus_{j=1}^{r} O(m_j))_{ss}.$$

Here $n_i \geq 0$ and $m_j \leq -2$ for all i, j.

Proof. One says that X has a Lefschetz pencil when it has such a morphism π; the generic fiber is a smooth curve X_K over $K = k(P^1)$. As $\pi_*O_X \cong O_{P^1}$, we have $R^0\pi_*F_p \cong F_p$. Let $F = R^1\pi_*F_p$. If η is the generic point, then $F_{\bar{\eta}} \cong H^1(X_{K_S}, F_p)$ where $X_{K_S} = X_K \otimes_K K_S$. Also $H^1(X_{K_S}, F_p) \cong H^1(X_{\bar{K}}, F_p)$ where $\bar{K}$ is the algebraic closure of K and $X_{\bar{K}} = X_K \otimes_K \bar{K}$. Since this latter cohomology group is known to be $Ker[H^1(X_{\bar{K}}, O_{X_{\bar{K}}}) \xrightarrow{FR-1} H^1(X_{\bar{K}}, O_{X_{\bar{K}}})]$, we see that $0 \leq dim_{F_p}(F_{\bar{\eta}}) \leq g_{X_{\bar{K}}}$ (= the genus of $X_{\bar{K}}$).

The value of this dimension depends on the p-rank of X_K as follows. Under the hypotheses, the Hochschild-Serre spectral sequence [15, III.2.21(b)] for $\Gamma_s = Gal(X_{K_s}/X_K)$ can be replaced by a corresponding spectral sequence [15, III.2.20] for Γ. The associated exact sequence of low degree terms becomes

$$H^1(\Gamma, F_p) \to H^1(X_K, F_p) \to H^1(X_{\bar{K}}, F_p)^{\Gamma} \to H^2(\Gamma, F_p) \to H^2(X_K, F_p)$$
$$\to H^1(\Gamma, H^1(X_{\bar{K}}, F_p)).$$

The first term of the exact sequence is $Hom(\Gamma_{ab}, F_p)$. The last term is $Hom(\Gamma_{ab}, F_p)^{\sigma_{\bar{K}}}$ since the F_p-dimension of $H^1(X_{\bar{K}}, F_p)$ is $\sigma_{\bar{K}}$. The condition that p does not divide the order of Γ_{ab} means that each of these groups is trivial. Now the assumption that X_K has non-zero p-rank will imply that $H^1(X_{\bar{K}}, F_p)$ is not trivial and $dim_{F_p}(F_{\bar{\eta}}) > 0$. From now on, we assume X_K is not supersingular, i.e. has non-zero p-rank.

We will need to know the cohomology of P^1 with coefficients in F. The sheaf $R^1\pi_* F_p$ is an étale group scheme. In fact, it is the étale core of $R^1\pi_* O_X$, which has a p-Lie algebra structure induced by the obvious p-linear morphism. As $(R^2\pi_* F_p)_{\bar{s}} \cong H^2(X_{\bar{s}}, F_p) = 0$ for all $s \in P^1$ ($X_{\bar{s}}$ is a curve), we see that $R^2\pi_* F_p = 0$. Now the Leray spectral sequence and the associated exact sequence of low degree terms show that $H^i(X, F_p) \cong H^{i-1}(P^1, F)$ for $i = 1, 2$. It follows that $\chi(X, F_p) = 1 - \chi(P^1, F)$.

The sheaf $R^1\pi_* O_X$ is locally free of finite rank on P^1, hence it decomposes into a direct sum of invertible sheaves $O(n)$. Index so that s indices n_i are ≥ 0, t indices $l_k = -1$, and r indices m_j are ≤ -2. Then $H^0(P^1, R^1\pi_* O_X) \cong \bigoplus_{i=1}^{s} H^0(P^1, O(n_i))$ and $H^1(P^1, R^1\pi_* O_X) \cong \bigoplus_{j=1}^{r} H^1(P^1, O(m_j))$. Since $R^0\pi_* O_X \cong \pi_* O_X \cong O_{P^1}$ and $R^2\pi_* O_X = 0$, the Leray spectral sequence and the associated exact sequence of low degree terms again show that

$$H^i(X_{et}, O_X) \cong H^{i-1}(P^1, R^1\pi_* O_X)$$

for $i = 1, 2$. Corollary 1.2 now shows that

$$\chi(P^1, F) = dim_k H^0(P^1, \bigoplus_{i=1}^{s} O(n_i))_{ss} - dim_k H^1(P^1, \bigoplus_{j=1}^{r} O(m_j))_{ss}.//$$

Remark. Without some additional hypothesis, it does not seem likely that this result can be made more precise. For other calculations over P^1, see Corollary 3.5 and the examples in Section III.

Section III. F_q-Vector Schemes

As an application of 2.4, we consider the cohomology of a smooth, proper curve with coefficients in a constructible sheaf whose generic stalk has rank $q = p^r$ ($r \geq 1$) corresponding to a finite étale F_q-vector scheme over the generic point. We begin with a general discussion of F_q-vector schemes over the generic point of an integral scheme X and height 1 schemes of F_q-vectors over X. We continue with the cohomology calculation above, including $X = P^1$, and finish with some examples over P^1.

Let X be an integral scheme with $K = k(X)$ and $\eta = Spec\ K$ the generic point. We want to consider F_q-vector schemes over η, where $q = p^r$, $r \geq 1$. Raynaud [19] showed that any finite étale F_q-vector scheme over η has the form $Spec\ (K[T_1, \ldots, T_r]/(T_1^p - \alpha_1 T_2, \ldots, T_r^p - \alpha_r T_1))$, $\alpha_i \in K^*$. Over any base scheme of characteristic p, F_q-vector schemes are classified by giving r invertible sheaves L_i and morphisms $a_i\colon L_{i+1} \to L_i^p$, $b_i\colon L_i^p \to L_{i+1}$

of O_X-modules, with $b_i \circ a_i$ the 0-morphism of L_{i+1}. (Here L_i^p denotes p-fold tensor product.) Equivalently, we can interpret a_i as an element of $\Gamma(X, L_i^p \otimes_{O_X} L_{i+1}^{-1})$ and b_i as an element of $\Gamma(X, L_{i+1} \otimes_{O_X} L_i^{-p})$, with $b_i \otimes a_i$ $(= a_i \otimes b_i)$ the 0-section of $\Gamma(X, O_X)$. We will use the notation $G^{\oplus L_i}_{a_i;\ b_i}$ for these group schemes.

If S is any X-scheme, then the S-valued points of $G = G^{\oplus L_i}_{a_i;\ b_i}$ can be viewed as the set of all $(x_1, \dots, x_r) \in \bigoplus_{i=1}^r \Gamma(S, L_i \otimes_{O_X} O_S)$ such that $x_i^p = a_i \otimes x_{i+1}$. When G is étale, each $b_i = 0$ since a_i is invertible. The Cartier dual of G is $G^{\oplus L_i^{-1}}_{b_i;\ a_i}$.

We need to analyze the Frobenius morphism $F\colon G \to G^{(p)}$. From [10, 6.1], one knows that for any locally-free sheaf E of O_X-modules over a scheme X of characteristic p, there is a functor $\pi\colon E \to E^{(p)}$ corresponding to replacing sections of O_X by their p-th powers. For an invertible sheaf L, it maps L to L^p; it also commutes with direct sums. Hence, it will map $\bigoplus L_i$ to $\bigoplus L_i^p$. It follows that $G^{(p)} = G^{\oplus L_i^p}_{a_i^p;\ b_i^p}$. Now, we see that the Frobenius is given by $F(x_1, \dots, x_r) = (x_1^p, \dots, x_r^p) = (a_1 \otimes x_2, \dots, a_r \otimes x_1)$. As the Verschiebung is the Frobenius morphism of G^D, the Verschiebung morphism $V\colon G^{(p)} \to G$ is given by $V(x_1', \dots, x_r') = (b_r \otimes x_r', b_1 \otimes x_1', \dots, b_{r-1} \otimes x_{r-1}')$.

When the Frobenius is the 0-morphism, each $a_i = 0$ and the b_i are arbitrary. The p-Lie algebra of $G = G^{\oplus L_i}_{0;\ b_i}$ is $\underline{Lie}(G) = \bigoplus_{i=1}^r L_i$ and the p-th power map sends $f = (f_1, \dots, f_r)$ to $(b_r \otimes f_r^p, b_1 \otimes f_1^p, \dots, b_{r-1} \otimes f_{r-1}^p)$ for $f \in \underline{Lie}(G)$. This identifies the p-th power map for height 1 schemes of F_q-vectors.

Remark. For such schemes, the p-Lie algebra is thus decomposable into a direct sum of invertible sheaves. It might prove interesting to study indecomposable p-Lie algebras.

Let $A^D = G^{\oplus L_i}_{a_i';\ 0}$ extend $G^{K^r}_{\alpha_i^{-1};\ 0}$, which equals *Spec* $(K[T_1, \dots, T_r]/(T_1^p - \alpha_1^{-1}T_2, \dots, T_r^p - \alpha_r^{-1}T_1))$. Let G be the étale core of $A = G^{\oplus L_i^{-1}}_{0;\ a_i'}$. If N_i is the invertible sheaf $L_i^p \otimes L_{i+1}^{-1}$, then the sections $a_i' \in \Gamma(X, N_i)$ give rise to sections $(a_i')_x \in \Gamma(x, (N_i)_x) \cong (N_i)_x$. The morphism $([\mathrm{p}] - 1)_x$ induced by base change is given by $(y_1, \dots, y_r) \mapsto ((a_r')_x \otimes y_r^p - y_1, (a_1')_x \otimes y_1^p - y_2, \dots, (a_{r-1}')_x \otimes y_{r-1}^p - y_r)$. Viewing $\underline{Lie}(A) = \bigoplus L_i^{-1}$ as a vector group, the exact sequence of 1.1 becomes

$$0 \to G_x \to Spec\ (k'[T_1, \dots, T_r]) \xrightarrow{(P_1, \dots, P_r)} Spec\ (k'[T_1, \dots, T_r]) \to 0.$$

Here $P_i(T_{i-1}, T_i) = c_{i-1}T_{i-1}^p - T_i$, with c_i the element of $k' = k(x)$ corresponding to $(a_i')_x$. If m_x is the maximal ideal of O_x, then c_i is the image of $(a_i')_x$ in $(N_i)_x/m_x(N_i)_x \cong k'$. It follows that $G_x \cong Spec\ (k'[T_1, \dots, T_r]/(c_1T_1^p - T_2, \dots, c_{r-1}T_{r-1}^p - T_r, c_rT_r^p - T_1))$.

The vanishing of a given c_i is equivalent to $(a_i')_x \in m_x(N_i)_x$. If any $c_i = 0$, then $G_x \cong Spec\, k'$. If all c_i are nonzero, then $G_x \cong G^{(k')^r}_{c_i^{-1};\ 0} \cong (F_q)_{k'}$.

Let W be the open set of all $x \in X$ such that $(a'_i)_x \notin m_x(N_i)_x$ for all i. We see that $G_{|W}$ is a finite étale F_q-vector scheme and $G_{|X-W}$ is trivial.

Finally, we want to establish that $G^{K^r}_{\alpha_i;\ 0} \cong \underline{Hom}_{K-grp}(G^{K^r}_{\alpha_i^{-1};\ 0}, F_p)$. Calling the first G and the second H, we show there is a nondegenerate pairing $G \times H \to F_p$; in fact, it need only be exhibited for global sections over K_s. Here $G(K_s)$ is the set of all $(b_1, \dots, b_r) \in K_s^r$ such that $b_i^p = \alpha_i b_{i+1}$ and $H(K_s)$ is the set of all $(d_1, \dots, d_r) \in K_s^r$ which satisfy $d_i^p = \alpha_i^{-1} d_{i+1}$. Note that $(b_i d_i)^p = b_{i+1} d_{i+1}$. Now, define the pairing by $(b_1, \dots, b_r) \times (d_1, \dots, d_r) \mapsto b_1 d_1 + \cdots + b_r d_r$. We check that $(b_1 d_1 + \cdots + b_{r-1} d_{r-1} + b_r d_r)^p = b_2 d_2 + \cdots + b_r d_r + b_1 d_1 = b_1 d_1 + \cdots + b_r d_r \in F_p$. As $b_i^q = \alpha_i^{p^{r-1}} \cdots \alpha_r^{p^{i-1}} \cdots \alpha_{i-1} b_i$, d_i^q is given by a similar formula involving α_i^{-1}, and the values of all other b_j (respectively d_j) are completely determined by a single b_i (respectively d_i), this is just the pairing $F_q \times F_q \to F_p$ of dual spaces, which is clearly nondegenerate.

We already know from 1.4 (and the Appendix) some conditions under which a finite étale K-scheme of F_q-vectors extends to a finite flat group scheme on all of X. We want this extension to be a F_q-vector scheme. Although 1.4 applies to curves, the results of the Appendix do not. We will prove a more specialized lemma that includes curves.

Lemma 3.1. *The finite étale K-group scheme of F_q-vectors $G^{K^r}_{\alpha_i;\ 0}$ extends to a finite flat F_q-vector scheme $G^{\oplus L_i}_{a_i;\ 0}$ on any quasi-compact integral scheme X with an ample divisor.*

Proof. Given X, it is possible to find r ample divisors $D'_1, \dots, D'_r$ such that $pD'_i - D'_{i+1}$ is ample for $i = 1, \dots, r$. We choose positive integers $m_1 \geq m_2 \geq \dots \geq m_r$ with $m_r > [\frac{m_1}{p}]$ and set $D'_i = m_i D$ for the ample divisor D. Define r invertible sheaves $L_{i,1} = L(D'_i)$. Then $L^p_{i,1} \otimes L^{-1}_{i+1,1} = L(pD'_i - D'_{i+1})$ is ample for all i. As in the proof of [12, 2.5], the rational function $\alpha_i \in K^*$ extends to a global section a_i of some tensor power of $L^p_{i,1} \otimes L^{-1}_{i+1,1}$, say $L^{pn_i}_{i,1} \otimes L^{-n_i}_{i+1,1}$. Taking $N = lcm(n_1, \dots, n_r)$, we get invertible sheaves $L_i = L^N_{i,1}$ such that a_i is a global section of $L^p_i \otimes L^{-1}_{i+1}$. Thus, we have morphisms $a_i \colon L_{i+1} \to L^p_i$. This determines a finite flat scheme of F_q-vectors $G^{\oplus L_i}_{a_i;\ 0}$ which extends the finite étale K-scheme $G^{K^r}_{\alpha_i;\ 0}$, as desired.//

Remark. If one assumes that there is a finite flat extension it is possible to prove the following proposition which gives an alternative to [12, 2.5] (and 3.1) for curves.

Proposition 3.2. *If a finite flat extension to all of X exists, then any finite étale K-scheme of F_q-vectors extends to a finite flat F_q-vector scheme on a regular irreducible scheme X of dimension 1.*

Proof. If X satisfies the given hypotheses, then the arguments of [19, Sections 2 and 3] and [3, 2.8] can be extended to show there is a maximal, finite, flat extension over X and that the F_q-vector structure of the generic fiber extends over this whole group scheme. Hence we may assume the extension has the form $G^{\oplus L_i}_{a_i;\ b_i}$. The form of the generic fiber will imply that all the sections a_i are nonzero and all the sections b_i are 0. If any section a_i is zero, so is the corresponding rational function α_i; similarly, a nonzero section b_i would give a rational function $\beta_i \neq 0$. Both contradict that $\alpha_i\beta_i = 0$ with $\alpha_i \neq 0$ for all i. We conclude that the extension has the desired form.//

Assume for the remainder of this section that X is a smooth proper curve over a field k. Then $L_i = L(D_i)$ for some divisor D_i on X. The requirement that a_i be a nonzero global section of $L_i^p \otimes L_{i+1}^{-1}$ implies that $pD_i \geq D_{i+1}$ for all i. Hence $qD_i \geq D_i$, i.e. $(q-1)D_i \geq 0$, and so $D_i \geq 0$.

Now, let F be a constructible sheaf of F_p-modules on X_{et} such that $G^{K^r}_{\alpha_i;\ 0}$ is the finite étale K-group scheme corresponding to $F_{\bar{\eta}}$, where $K = k(X)$. We denote the genus of X by g_X. As in 3.1, let $A^D = G^{\oplus L_i}_{a'_i;\ 0}$ extend $G^{K^r}_{\alpha_i^{-1};\ 0}$ and G be the étale core of $A = G^{\oplus L_i^{-1}}_{0;\ a'_i}$.

Theorem 3.3. *Let $L_i = L(D_i)$ for $pD_i > D_{i+1}$, all i. Then $H^0(X_{et}, G) = H^2(X_{et}, G) = 0$ and $dim_{F_p} H^1(X_{et}, G) = \sigma$, where σ is the stable rank of the matrix B corresponding to the induced homomorphism $H^1([p])$.*

Proof. Since $\bigoplus_{i=1}^r L_i^{-1}$ has no global sections and $\bigoplus H^2(X_{et}, L_i^{-1}) = 0$ for the curve X, we need only analyze $\bigoplus H^1(X_{et}, L_i^{-1})$. The Riemann-Roch theorem shows that $H^1(X_{et}, L_i^{-1})$ has k-dimension g_X+deg $D_i - 1$. Since $\underline{Lie}(A) = \bigoplus L_i^{-1}$, we apply Corollary 1.2 to determine $H^1(X_{et}, G)$. The value for σ follows from the discussion before 1.2.//

Theorem 3.4. *Let $L_i = L(D_i)$ for $pD_i \geq D_{i+1}$ and $a'_i \in \Gamma(X, L_i^p \otimes L_{i+1}^{-1})$, all i. Set $a' = a'_1 \otimes \cdots \otimes a'_r$ and $L = L_1 \otimes \cdots \otimes L_r$. Let $n(a')$ be the cardinality of the support of the divisor of zeroes $(a')_0$ of $a' \in \Gamma(X, L^{p-1})$. Then*

$$\chi(X_{et}, F) = \chi(X_{et}, G) + r \cdot n(a') - \sum_{x \in X^0} t_x(F).$$

Proof. Since $a'_i \in \Gamma(X, L_i^p \otimes L_{i+1}^{-1})$, we find that a' is a global section of $(L_1^p \otimes L_2^{-1}) \otimes (L_2^p \otimes L_3^{-1}) \otimes \cdots \otimes (L_r^p \otimes L_1^{-1}) \cong L_1^{p-1} \otimes \cdots \otimes L_r^{p-1} \cong L^{p-1}$. The set of all $x \in X$ satisfying $a'_x = (a'_1)_x \otimes \cdots \otimes (a'_r)_x \in m_x L_x^{p-1}$ is the support of the divisor of zeroes $(a')_0$ of $a' \in \Gamma(X, L^{p-1})$. By the discussion before Lemma 3.1, we know $G_x = Spec\ k(x)$ when any $(a'_i)_x \in m_x(L_i^p \otimes L_{i+1}^{-1})_x$ and hence $G_{\bar{x}} = 0$.

Moreover, let R be a local ring with maximal ideal m and A, B be free R-modules. The images of $m \otimes (A \otimes B)$, $(m \otimes A) \otimes B$, and $A \otimes (m \otimes B)$ are the submodules $m(A \otimes B)$, $mA \otimes B$, and $A \otimes mB$ respectively of $A \otimes B$ under the isomorphisms of $R \otimes (A \otimes B)$, $(R \otimes A) \otimes B$, and $A \otimes (R \otimes B)$

respectively with $A \otimes B$. Hence $m(A \otimes B) \cong mA \otimes B \cong A \otimes mB$. It follows that $(a')_0 = \bigcup_{i=1}^{r} (a'_i)_0$. Thus, $r \cdot n(a') = \sum_{x \in X^0} dim_{F_p} G_{\bar\eta} - dim_{F_p} G_{\bar x} = \sum_{x \in X^0} t_x(G)$. The result follows by Corollary 2.4.//

We now specialize to the case $X = P^1$, so that more explicit calculations can be made. Let F be a constructible sheaf of F_p-modules such that $F_{\bar\eta} = G^{K^r}_{\alpha_i;\,0}(K_s)$, where $K = k(P^1)$ and $\alpha_i \in K^*$ for all i. Choose homogeneous coordinates X_0, X_1 on P^1 and set $T = X_1/X_0$. Then $K = k(T)$. We assume that $\alpha_i^{-1} \in k[T]$ for all i.

Let $L_i = O(n_i)$ with $pn_i \geq n_{i+1}$. We know that $qn_i \geq n_i$ and so $n_i \geq 0$. The rational function α_i^{-1} extends to a global section $a'_i \in \Gamma(X, L_i^p \otimes L_{i+1}^{-1}) = \Gamma(X, O(pn_i - n_{i+1}))$. Set $M_i = pn_i - n_{i+1}$ for all i. Then G will be the étale core of $A = G^{\oplus O(-n_i)}_{0;\, a'_i}$. We want to determine $\chi(P^1, G)$.

Notation. If $M_i = pn_i - n_{i+1} > 0$, we write the global section a'_i of $O(M_i)$ as a homogeneous polynomial $a'_i = \sum_{j=0}^{M_i} d_{ij} X_0^{M_i - j} X_1^j$. Denote by $A(a'_i)$ the $(n_{i+1} - 1) \times (n_i - 1)$ matrix

$$\begin{pmatrix} d_{i,\ p-1} & d_{i,\ 2p-1} & \dots & d_{i,\ (n_i-1)p-1} \\ d_{i,\ p-2} & d_{i,\ 2p-2} & \dots & d_{i,\ (n_i-1)p-2} \\ \vdots & \vdots & & \vdots \\ d_{i,\ p+1-n_{i+1}} & d_{i,\ 2p+1-n_{i+1}} & \dots & d_{i,\ (n_i-1)p+1-n_{i+1}} \end{pmatrix}.$$

Remark. If $N > 0$, then any global section of $a' = \sum_{j=0}^{N} d_j X_0^{N-j} X_1^j$ of $O(N)$ uniquely determines a polynomial $\alpha^{-1} = \sum d_j T^j$ in $k[T]$ of degree $\leq N$, via the isomorphism $O(N) \otimes K \to K$ given by $X_0^N \to 1$.

Theorem 3.5 (Same notation as 3.4, except that $X = P^1$ and $L_i = O(n_i)$ with all $n_i \geq 0$ but $\neq 1$)**.**

$$\chi(P^1, F) = d - \sigma(a') + r \cdot \# - \sum_{x \in X^0} t_x(F),$$

where $d = r$ if some $n_i = 0$ and 0 otherwise and $\sigma(a')$ is the stable rank of a matrix of the form

$$B = \begin{pmatrix} 0 & \dots & 0 & B_r \\ B_1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & B_{r-1} & 0 \end{pmatrix}.$$

Here $\#$ is the number of distinct zeroes and poles of the polynomial α^{-1} determined by a', less 1 when $0 < deg\ \alpha^{-1} = M(p-1)$, for $M = \sum_{i=1}^{r} n_i$.

Proof.

Case 1. If some $n_i = 0$, then $pn_i \geq n_{i+1} \geq 0$ and we have $n_{i+1} = 0$. It follows that all $n_i = 0$ in this case. Hence, $M_i = pn_i - n_{i+1} = 0$ for all i and a_i' lies in the ground field. We have $\underline{Lie}(A) = \bigoplus_{i=1}^r O_{P^1}$ with $\bigoplus_{i=1}^r H^0(O_{P^1}) \cong k^r$. The matrix B of the induced homomorphism $H^0([p])$ with respect to the canonical basis is

$$B = \begin{pmatrix} 0 & \dots & 0 & a_r' \\ a_1' & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & a_{r-1}' & 0 \end{pmatrix}.$$

Moreover, $a' = a_1' \otimes a_2' \otimes \cdots \otimes a_r'$ itself lies in the ground field, so $n(a') = 0$ and $\#$ does also. The stable rank of B is the rank of the matrix $B\,B^{(p)} \cdots B^{(p^{r-1})}$. Since all the $a_i' \neq 0$, it is clear that the matrix B (and hence the matrices $B^{(p^t)}$) is invertible. It follows that the previous product of matrices is invertible and so has rank r. This gives the desired value of d.

From now on, we assume all $n_i > 0$. More specifically, suppose $n_i \geq 2$ for all i. Theorem 3.3 shows that $\chi(P^1, G) = -\sigma$, where σ is the stable rank of the matrix B corresponding to the induced homomorhism $H^1([p])$. We want to study B with respect to the canonical bases of $\bigoplus H^1(O(-n_i))$. Note that $H^1([p])$ maps $H^1(O(-n_i))$ to $H^1(O(-n_{i+1}))$.

Case 2. Suppose $M_i > 0$ for all i. The k-vector space $H^0(O(n_i-2))$ has a canonical basis $X_0^{n_i-2}, X_0^{n_i-3}X_1, \dots, X_0^{n_i-2-j}X_1^j, \dots, X_1^{n_i-2}$. The dual space $H^1(O(-n_i))$ has dual basis $X_0^{-n_i+1}X_1^{-1}, X_0^{-n_i+2}X_1^{-2}, \dots, X_0^{-n_i+1+j}$ $X_1^{-1-j}, \dots, X_0^{-1}X_1^{-n_i+1}$. For $l=0, \dots, n_i-2$, let $w_{l+1,i} = X_0^{-n_i+1+l}X_1^{-1-l}$. The induced homomorphism $H^1([p])$ is given by $w_{l+1,\ i} \mapsto a_i' \otimes w_{l+1,\ i}^p$ for $i = 1, \dots, r$. Calculating the image, we get

$$\begin{aligned} a_i' \otimes w_{l+1,\ i}^p &= \sum_{j=0}^{M_i} d_{i,\ j} X_0^{M_i-j} X_1^j \otimes X_0^{-pn_i+p+pl} X_1^{-p-pl} \\ &= \sum_{j=0}^{M_i} d_{i,\ j} X_0^{pn_i-n_{i+1}-j-pn_i+p(l+1)} X_1^{j-p(l+1)} \\ &= \sum_{j=0}^{M_i} d_{i,\ j} X_0^{-n_{i+1}-j+p(l+1)} X_1^{j-p(l+1)} \\ &= \sum_{s=0}^{n_{i+1}-2} d_{i,\ p(l+1)-(s+1)} X_0^{-n_{i+1}+1+s} X_1^{-1-s}, \end{aligned}$$

where we have made the substitution $j = p(l+1) - (s+1)$ to get the last summation.

It follows that the matrix of $H^1([p])\colon H^1(O(-n_i)) \to H^1(O(-n_{i+1}))$ with respect to the canonical bases of each is $B_i = A(a_i')$. Thus, we have a decomposition of the matrix B of $H^1([p])$ with respect to the canonical bases of $\bigoplus_{i=1}^r H^1(O(-n_i))$ in the form

$$B = \begin{pmatrix} 0 & \dots & 0 & B_r \\ B_1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & B_{r-1} & 0 \end{pmatrix}.$$

Forming $a' = a'_1 \otimes a'_2 \otimes \cdots \otimes a'_r$, we get a global section of $O(M(p-1))$, where $M = \sum_{i=1}^r n_i$. The global section a' uniquely determines a nonhomogeneous polynomial $\alpha^{-1} \in k[T]$. As in the proof of [12, 3.1], we can interpret $n(a')$ as the number of distinct zeroes of α^{-1}, regarded as a rational function, together with the point at infinity, except when $0 < deg\ \alpha^{-1} = M(p-1)$.

The stable rank of the matrix B is the rank of the matrix $B\,B^{(p)} \cdots B^{(p^{d-1})}$, where $d = dim_k \bigoplus_{i=1}^r H^1(O(-n_i)) = \sum_{i=1}^r (n_i - 1) = M - r$. Each of the matrices $B^{(p^t)}$ has a decomposition exactly like that of B, with B_i replaced by $B_i^{(p^t)}$.

Case 3. Suppose some $M_i = pn_i - n_{i+1} = 0$, but not all. (If all $M_i = 0$, we are back in Case 1.) Then a'_i lies in the ground field and the induced homomorphism maps $w_{l+1,\ i}$ to $a'_i \otimes w^p_{l+1,\ i} = a'_i X_0^{-pn_i+p+pl} X_1^{-p-pl} = a'_i X_0^{-n_{i+1}+p(l+1)} X_1^{-p(l+1)}$. Here $l = 0, \dots, n_i - 2$ as before and $p(l+1) - 1 = p-1, 2p-1, \dots, (n_i-1)p - 1 = n_{i+1} - p - 1$. Again, we get matrices B_i of size $(n_{i+1}-1) \times (n_i - 1)$. Each matrix B_i has the constant entries a'_i in the positions $(p-1,\ 1), (2p-1, 2), \dots, (n_{i+1} - p - 1, n_i - 1)$ and 0 elsewhere. The matrix B still has a decomposition of the same form as in Case 2.//

Remark. The case of some $n_i = 1$ has been omitted from the theorem because it is more complicated to characterize. Suppose some $n_i = 1$, but n_{i-1} and n_{i+1} are both ≥ 2. Both cohomology groups $H^0(O(-n_i))$ and $H^1(O(-n_i))$ vanish. The global sections a'_i and a'_{i-1} of $O(M_i)$ (and $O(M_{i-1})$ respectively) determine a morphism $O(-n_{i-1}) \to O(-n_{i+1})$. The induced homomorphism on H^1 sends $w_{l+1,\ i-1}$ to $a'_i \otimes a'^p_{i-1} \otimes w^{p^2}_{l+1,\ i-1}$.

Examples.

We maintain the notation of Theorem 3.5, i.e. $X = P^1$ and $K = k(P^1)$. We choose homogeneous coordinates X_0,X_1 on P^1 and set $T = X_1/X_0$. Let $p = 7$ and $q = p^2$. We want to construct a family of examples using polynomials over F_7 to illustrate the theorem.

To keep the calculations manageable, we restrict to the case $r = 2$. More precisely, we assume that the generic stalk of the constructible sheaf F corresponds to $G^{K^2}_{\alpha_1,\alpha_2;\ 0,0}$ for $\alpha_1,\ \alpha_2 \in K^*$. In addition, we have that $A^D = G^{O(n_1)\oplus O(n_2)}_{a'_1,a'_2;\ 0,0}$ extends the finite étale K-scheme of F_q-vectors $G^{K^2}_{\alpha_1^{-1},\alpha_2^{-1};\ 0,0}$ where α_1^{-1}, $\alpha_2^{-1} \in k[T]$. We will use the étale core G of $A = G^{O(-n_1)\oplus O(-n_2)}_{0,0;\ a'_1,a'_2}$ and compute $\chi(P^1, F)$.

The matrix B required for the calculation of $\sigma(a')$, where $a' = a'_1 \otimes a'_2$, will have the form $B = \begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix}$, where $B_i = A(a'_i)$. Since the coefficients lie in F_7, we have $B^{(p)} = B^{(7)} = B$ and so we must compute the rank of B^2. Two cases will be considered: 1) both α_1^{-1}, α_2^{-1} are squares of polynomials of degree 6; 2) both are squares of polynomials of degree 5.

A. Degree 6

The rational functions α_1, α_2 will have the form $\alpha_i = [h(T)]^{-2}$, where $h(T) = \prod_{j=1}^{6} (T - b_{ij})$, with the b_{ij} distinct elements of F_7 $(i = 1, 2)$. In the following table, we have $\alpha_i^{-1} = T^{12} + d_{i,11}T^{11} + \cdots + d_{i,1}T + d_{i,0}$. We designate the corresponding functions, i.e. corresponding to the distinct rows of the table, by $\gamma_1^{-1}, \ldots, \gamma_7^{-1}$.

Degree 6 Table

b_1	b_2	b_3	b_4	b_5	b_6	d_{11}	d_{10}	d_9	d_8	d_7	d_6	d_5	d_4	d_3	d_2	d_1	d_0
0	1	2	3	4	5	5	3	3	5	1	5	3	3	5	1	0	0
0	1	2	3	4	6	3	5	3	3	4	5	6	5	5	2	0	0
0	1	2	3	5	6	1	6	4	6	5	5	2	6	2	4	0	0
0	1	2	4	5	6	6	6	3	6	2	5	5	6	5	4	0	0
0	1	3	4	5	6	4	5	4	3	3	5	1	5	2	2	0	0
0	2	3	4	5	6	2	3	4	5	6	5	4	3	2	1	0	0
1	2	3	4	5	6	0	0	0	0	0	5	0	0	0	0	0	1

1. $n_1 = n_2 = 2$; $M = n_1 + n_2 = 4$

The polynomial functions α_i^{-1} extend to global sections a'_i of $O(2p - 2) = O(12)$. The components of the 2×2 matrices B are 1×1 submatrices $B_i = A(a'_i) = (d_{i,6})$. We consider subcases corresponding to the pairs (γ_1^{-1} row, γ_2^{-1} row), (γ_3^{-1} row, γ_4^{-1} row), (γ_5^{-1} row, γ_6^{-1} row), and (γ_7^{-1} row, γ_7^{-1} row). In all four subcases, the matrix $B = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix} = B^{(p)}$ is invertible. Hence $B\,B^{(p)} = B^2$ has rank 2 and $\sigma(a') = 2$. For the first three subcases, the value of # is always $7 = 8 - 1$ since there are seven distinct zeroes and poles of $\alpha^{-1} \stackrel{\text{def}}{=} \alpha_1^{-1}\alpha_2^{-1}$ (excluding ∞ for each subcase, as $deg\ \alpha^{-1} = 4(6)$). For the fourth subcase, the value of # is $6 = 7 - 1$ (∞ is again excluded). Finally, $d = 0$ in each instance. We conclude that $\chi(P^1, F) = 12 - \sum_{x \in X^0} t_x(F)$ for the first three subcases and equals $10 - \sum_{x \in X^0} t_x(F)$ for the fourth.

2. $n_1 = 2$, $n_2 = 3$ (or vice versa)

This is impossible since $2p - 3 = 11 < 12$.

3. $n_1 = n_2 = 3$; $M = n_1 + n_2 = 6$

The α_i^{-1} extend to global sections a'_i of $O(3p - 3) = O(18)$. The components of the 4×4 matrix B are 2×2 submatrices $B_i = A(a'_i) =$

$\begin{pmatrix} d_{i,6} & d_{i,13} \\ d_{i,5} & d_{i,12} \end{pmatrix}$. Note that $d_{i,6} = 5$, $d_{i,12} = 1$, and $d_{i,13} = 0$ for all subcases. The matrix B is always invertible here. Hence $B\,B^{(p)} = B^2$ is as well and $\sigma(a') = 4$. In addition, $deg\ \alpha^{-1} = 24 \neq 6(6)$ and we see $\# = 8$ for the first three subcases and equals 7 for the fourth. As before, $d = 0$. The results are the same as in Case 1. (Of course, this is as it should be since the extension to a scheme of F_q-vectors need not be unique.)

B. Degree 5

If we mimic here the approach we took for degree 6, we get a table that has 21 rows, all of which have nonzero coefficient d_6. The results for $n_1 = n_2 = 2$ will predict all the possibilities (although $\sigma(a')$ and $\#$ can vary with n_1, n_2) and will be left to the reader.

Degree 5 Table

b_1	b_2	b_3	b_4	b_5	d_9	d_8	d_7	d_6	d_5	d_4	d_3	d_2	d_1	d_0
0	1	2	3	4	1	2	5	5	3	1	1	2	0	0
0	1	2	3	5	6	0	5	3	1	0	1	4	0	0
0	1	2	3	6	4	0	2	5	5	0	3	1	0	0
0	1	2	4	5	4	4	5	6	6	1	4	4	0	0
0	1	2	4	6	2	1	5	3	5	1	2	1	0	0
0	1	2	5	6	0	4	0	5	0	2	0	2	0	0
0	1	3	4	5	2	0	2	6	3	0	5	2	0	0
0	1	3	4	6	0	1	0	6	0	2	0	4	0	0
0	1	3	5	6	5	1	2	3	2	1	5	1	0	0
0	1	4	5	6	3	0	5	5	2	0	4	1	0	0
0	2	3	4	5	0	2	0	3	0	2	0	1	0	0
0	2	3	4	6	5	0	5	6	4	0	2	2	0	0
0	2	3	5	6	3	4	2	6	1	1	3	4	0	0
0	2	4	5	6	1	0	2	3	6	0	6	4	0	0
0	3	4	5	6	6	2	2	5	4	1	6	2	0	0
1	2	3	4	5	5	3	3	5	1	5	3	3	5	1
1	2	3	4	6	3	5	3	3	4	5	6	5	5	2
1	2	3	5	6	1	6	4	6	5	5	2	6	2	4
1	2	4	5	6	6	6	3	6	2	5	5	6	5	4
1	3	4	5	6	4	5	4	3	3	5	1	5	2	2
2	3	4	5	6	2	3	4	5	6	5	4	3	2	1

As an alternative, we will consider the following three functions (which are the squares of polynomials of degree 5) namely, γ_1^{-1} as the first row, γ_2^{-1} as the last row, and $\gamma_3^{-1} = (T^5 - 1)^2 = (T^{10} - 2T^5 + 1) \equiv (T^{10} + 5T^5 + 1)\ (mod\ 7)$. The pairs will consist of (1st row, 1st row), (1st row, γ_3^{-1} row), (last row, γ_3^{-1} row) and (γ_3^{-1} row, γ_3^{-1} row). For variety, we choose $n_1 = 2, n_2 = 3$; $M = n_1 + n_2 = 5$.

The polynomial functions $\alpha_1^{-1},\ \alpha_2^{-1}$ extend to global sections $a_1',\ a_2'$ of $O(2p-3) = O(11)$ and $O(3p-2) = O(19)$, respectively. The 3×3 matrix

B has components $B_1 = A(a_1')\begin{pmatrix} d_{1,6} \\ d_{1,5} \end{pmatrix}$ of size 2×1 and $B_2 = A(a_2') = (d_{2,6}\ d_{2,13})$ of size 1×2. Specifically, we have

$$B = \begin{pmatrix} 0 & d_{2,6} & d_{2,13} \\ d_{1,6} & 0 & 0 \\ d_{1,5} & 0 & 0 \end{pmatrix}.$$

1. (1st row, 1st row)

The matrix is $B = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 0 & 0 \\ 3 & 0 & 0. \end{pmatrix}$ The matrix $B^2 \equiv \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $(mod\ 7)$ and we see that $\sigma(a') = 2$. The value of # is 6 as α^{-1} has six distinct zeroes and poles (including ∞, since $deg\ \alpha^{-1} = 20 \neq 5(6)$). As $d = 0$, we get $\chi(F) = 10 - \sum_{x \in X^0} t_x(F)$.

2. (1st row, γ_3^{-1} row)

The matrix is $B = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$. As B^2 is the 3×3 zero matrix, we have $\sigma(a') = 0$. Here $\# = 10$ as α^{-1} has ten distinct zeroes and poles (including ∞). We have $d = 0$. Thus, $\chi(F) = 20 - \sum_{x \in X^0} t_x(F)$.

3. (last row, γ_3^{-1} row)

This time the matrix is $B = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix}$. Again, B^2 is the zero matrix and $\sigma(a') = 0$. Now, we get $\# = 11$ as there are eleven distinct zeroes and poles (including ∞). Finally, $d = 0$ and $\chi(F) = 22 - \sum_{x \in X^0} t_x(F)$.

4. (γ_3^{-1} row, γ_3^{-1} row)

In this case, the only nonzero entry is $d_{1,5} = 5$, giving B^2 the zero matrix and $\sigma(a') = 0$. For #, we get 6, corresponding to the six distinct zeroes and poles (including ∞). We know that d is unchanged and so $\chi(F) = 12 - \sum_{x \in X^0} t_x(F)$.

Appendix. Étale Extension

Using the theory of the fundamental group and the notion of purity, we study the conditions under which an extension of a finite étale group scheme over an open subscheme U of an integral scheme X to a finite étale group scheme on all of X exists. If Z is the complement of U, then the concept of étale (respectively, homotopical) Z-depth allows us to prove a relative result along these lines.

We maintain the notation introduced at the end of Section I. So H will be a finite étale group scheme extending a finite étale K-group scheme H_1 to some open subscheme U of X, where X is an integral scheme with $K = k(X)$. We may assume U is the maximal open subscheme with this property. We want to study when $U = X$. More precisely, we study the homomorphism $j_*: \Pi_1(U, \bar{x}) \to \Pi_1(X, \bar{x})$ of fundamental groups induced by the open immersion $j: U \to X$, where $\bar{x}$ is a geometric point of U (see [4, V.6]).

Of course, the categories FET/U and FET/X are equivalent if and only if j_* is an isomorphism. Under the equivalence, group schemes/U will correspond to group schemes/X since the definition of a group object in a category is via morphisms.

Proposition A.1 (Notation as above)**.** *H extends to a finite étale group scheme over X in the following cases:*

1) U is an open subscheme whose complement Z has codimension ≥ 2 and all the local rings at points of Z are regular.

2) U is an open subscheme whose complement Z is a local set-theoretic complete intersection of codimension ≥ 3.

Proof. Both parts of this theorem follow from the theory on purity developed in [5, X.3].//

Let $depth_Z(F)$ denote the étale Z-depth of F and $hom.depth_Z(X)$ denote the homotopical Z-depth of X. We obtain a relative result.

Theorem A.2. *Let $f: X \to S$ be a regular morphism of locally noetherian schemes and Z a closed subset of X with $U = X - Z$. Suppose that for any point $s \in f(Z)$, one of the following conditions holds:*

a) codimension $(Z_s, X_s) \geq 2$;

b) codimension $(Z_s, X_s) \geq 1$ and $depth_s(S) \geq 1$;

c) $hom.depth_s(S) \geq 3$.

Then H extends to a finite étale group scheme over X. In fact, for any X' étale over X, let U' denote the restriction of U to X'. Then $H' = H \times_U U'$ extends to a finite étale group scheme over X'.

Proof. This follows from the theory developed in [5, XIV.1.20].//

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