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A Solvable Stochastic Control Problem in Real Hyperbolic Three Space II*

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Abstract

A stochastic optimal control problem is formulated and explicitly solved in a real hyperbolic space of dimension three. The problem is to control Brownian motion in this noncompact symmetric space by a drift vector field so that the controlled diffusion remains close to the origin. This problem modifies a previous explicitly solvable control problem of the author to provide a more enhanced formulation and solution. The stochastic control problem is solved by finding a smooth solution to the Hamilton-Jacobi equation.

1. Introduction

While a sizable amount of research in stochastic optimal control has been performed, there are only a relatively few examples of controlled diffusions where the optimal control is expressed explicitly as a function of the state. In [1] a stochastic optimal control problem is formulated and explicitly solved where the controlled diffusion is Brownian motion plus a drift control in the real hyperbolic three space $\mathbb{H}^3(\mathbb{R})$. This problem seems to be the first example of an explicitly solvable stochastic control problem for a controlled diffusion in a noncompact manifold with nonzero curvature. The objective is to control Brownian motion with a drift vector field so that it remains close to the origin of $\mathbb{H}^3(\mathbb{R})$. The cost functional uses the hyperbolic cosine function in a basic way because this function is an eigenfunction for the radial part of the Laplace-Beltrami operator on $\mathbb{H}^3(\mathbb{R})$. However, the control problem in [1] has three less than desirable properties. The cost functional is not zero at the origin with no control, the optimal control has a singularity at the origin and the finite time interval

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for the optimal control is not arbitrary. In this paper these three, somewhat undesirable, properties are eliminated though the controlled diffusion has the same form.

2. Preliminaries and Main Result

Real hyperbolic three space, $\mathbb{H}^3(\mathbb{R})$, is a noncompact symmetric space of rank one that is described as the following quotient of semisimple Lie groups

$$\mathbb{H}^3(\mathbb{R}) \simeq SL(2,\mathbb{C})/SU(2) \tag{1}$$

For the analysis of the stochastic control problem it is useful to have a geometric model for $\mathbb{H}^3(\mathbb{R})$. While various geometric models exist for $\mathbb{H}^3(\mathbb{R})$, the unit ball model

$$B_1(0) = \{y \in \mathbb{R}^3 : |y| < 1\}, \tag{2}$$

where $|\cdot|$ is the usual Riemannian metric in $\mathbb{R}^3$, is particularly convenient here. $B_1(0)$ with the Riemannian structure

$$ds^2 = 4(1-|y|^2)^{-2}(dy_1^2 + dy_2^2 + dy_3^2) \tag{3}$$

has constant sectional curvature -1. It is well known that any space of constant negative curvature is globally trivial, that is, it is globally diffeomorphic to Euclidean space.

Geodesic polar coordinates for $\mathbb{H}^3(\mathbb{R})$ at the origin, denoted by 0, are a useful coordinate system. These coordinates are defined by the map

$$\mathrm{Exp}_0 Y \longmapsto (r, \theta_1, \theta_2) \tag{4}$$

where $Y \in T_0\mathbb{H}^3(\mathbb{R})$,$r = |Y|_0$ with $|\cdot|_0$ the Riemannian metric at 0 and (θ_1, θ_2) are coordinates of the unit vector $Y/|Y|_0$. In these coordinates the Riemannian structure (3) is

$$ds^2 = dr^2 + (\sinh\, r)^2 d\sigma^2 \tag{5}$$

where $d\sigma^2$ is the Riemannian structure on the unit sphere in $T_0\mathbb{H}^3(\mathbb{R})$ (e.g. [4]).

The controlled diffusion process has the infinitesimal generator

$$\frac{1}{2}\Delta_{\mathbb{H}^3(\mathbb{R})} + u\frac{\partial}{\partial r} \tag{6}$$

where $\Delta_{\mathbb{H}^3(\mathbb{R})}$ is the Laplace-Beltrami operator on $\mathbb{H}^3(\mathbb{R})$. The cost functional for the stochastic control problem is

$$J(U) = E_{|Y(0)|_0} \int_0^T a \sinh^2 \frac{|Y(t)|_0}{2} + \left(\cosh^2 \frac{|Y(t)|_0}{2}\right) U^2(t)\, dt \tag{7}$$

where $(Y(t),\, t \in [0,T])$ is the controlled diffusion in geodesic polar coordinates with infinitesimal generator (6) and $a > 0$. For notational simplicity the dependence of this diffusion on the control has been suppressed. Since the cost functional with the control fixed is invariant under the action of the maximal compact subgroup $SU(2)$, that is, it is constant on each sphere with center 0 in the unit ball model, it suffices to consider the controlled diffusion projected to the positive Weyl chamber, $\{r \colon r \geq 0\}$, which has the infinitesimal generator

$$\frac{1}{2}\tilde{\Delta}_{\mathbb{H}^3(\mathbb{R})} + u\frac{\partial}{\partial r} = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + u\frac{\partial}{\partial r} \tag{8}$$

and satisfies the stochastic differential equation

$$\begin{aligned} dX(t) &= (\coth X(t) + U(t))dt + dB(t) \\ X(0) &= |Y(0)|_0 \end{aligned} \tag{9}$$

The differential operator $\tilde{\Delta}_{\mathbb{H}^3(\mathbb{R})}$ is called the radial part of the Laplace-Beltrami operator $\Delta_{\mathbb{H}^3(\mathbb{R})}$ [4].

An admissible control at time t is a Borel measurable function of $Y(t)$ such that the stochastic differential equation whose infinitesimal generator is (6) has one and only one strong solution. By the invariance of the cost functional under the action of the maximal compact subgroup $SU(2)$, it suffices to consider controls at time t that are Borel measurable functions of $X(t)$ so that the solution of the stochastic differential equation (9) has one and only one strong solution.

Initially it is necessary to verify that the stochastic control problem (6-7) is well posed, that is, there is at least one control that gives a finite value to $J(U)$ in (7).

Lemma. *Let $(\tilde{X}(t),\ t \geq 0)$ be the diffusion process on $\mathbb{R}_+$ with the infinitesimal generator $\frac{1}{2}\tilde{\Delta}_{\mathbb{H}^3(\mathbb{R})}$, that is, the unique strong solution of* (9) *with $U(t) \equiv 0$. Then*

$$E_{X(0)} \int_0^T a \sinh^2 \frac{\tilde{X}(t)}{2} dt < \infty \tag{10}$$

This lemma is easily verified so the proof is not included here (e.g., [2]).

The solution of the stochastic control problem (6-7) is given now.

Theorem. *The stochastic optimal control problem described by* (6-7) *has an optimal control U^* that is*

$$U^*(s,y) = -\frac{1}{2}g(s)\tanh |y|_0 \tag{11}$$

where $s \in [0,T]$, y is given in geodesic polar coordinates (4) *and g is the unique positive solution of the Riccati differential equation*

$$\begin{aligned} &\frac{dg}{ds} + \frac{3}{2}g - \frac{1}{4}g^2 + a = 0 \\ &g(T) = 0 \end{aligned} \tag{12}$$

Proof. It is well known (e.g., [3]) that the Hamilton-Jacobi or dynamic programming equation for the control of a diffusion process is

$$0 = \frac{\partial W}{\partial s} + \min_{v \in U}[A^v(s)W + L(s,x,v)] \tag{13}$$

where A^v is the infinitesimal generator of the diffusion using the control v, and L is the integrand of the cost functional. To apply a verification theorem (p. 139 [3]) to the problem here, it is required that the solution W of (13) with the boundary condition $W(s,y) = 0$ for $(x,y) \in \{T\} \times \mathbb{H}^3(\mathbb{R})$ is $C^{1,2}[(0,T) \times \mathbb{H}^3(\mathbb{R})]$ and continuous on $[0,T] \times \mathbb{H}^3(\mathbb{R})$.

Since the integrand of the cost functional as a function of the state with the control fixed is invariant under the action of the maximal compact subgroup $SU(2)$, the Hamilton-Jacobi equation (13) can be reduced to the radial part of the infinitesimal generator (6) as

$$\begin{aligned} 0 = &\frac{\partial W}{\partial s} \\ &+ \min_{v \in \mathbb{R}} \left[\frac{1}{2}\frac{\partial^2 W}{\partial r^2} + \coth r \frac{\partial W}{\partial r} + v\frac{\partial W}{\partial r} + a \sinh^2 \frac{r}{2} + v^2 \cosh^2 \frac{r}{2}\right] \end{aligned} \tag{14}$$

Performing the minimization in (14), it is clear that the candidate for an optimal control is

$$U^*(s,r) = \frac{-1}{2\cosh^2 \frac{r}{2}} \frac{\partial W(s,r)}{\partial r} \tag{15}$$

Assume a solution W of (14) as

$$W(s,r) = g(s)\sinh^2 \frac{r}{2} + h(s) \tag{16}$$

It is useful to recall that

$$\sinh^2 \frac{r}{2} = \frac{1}{2}(\cosh r - 1).$$

Substitute (16) in (14) to obtain

$$\begin{aligned} 0 &= g' \sinh^2 \frac{r}{2} + h' + \frac{3}{4}g \cosh r \\ &\quad - \frac{1}{4}g^2 \sinh^2 \frac{r}{2} + a \sinh^2 \frac{r}{2} \\ &= \sinh^2 \frac{r}{2}\left[g' + \frac{3}{2}g - \frac{1}{4}g^2 + a\right] \\ &\quad + h' + \frac{3}{4}g. \end{aligned} \tag{17}$$

If g in (17) satisfies (12) and

$$\begin{aligned} h' + \frac{3}{4} g &= 0 \\ h(T) &= 0 \end{aligned} \tag{18}$$

then (16) is a smooth solution to (14) that satisfies the boundary condition $W(s,y) = 0$ for $(s,y) \in \{T\} \times \mathbb{H}^3(\mathbb{R})$. The control (15) is an admissible (optimal) control provided the stochastic differential equation (9) with this control has one and only one strong solution in $[0,T]$.

Since the drift term of the stochastic differential equatin (9) is locally smooth on $(0,\infty)$ with the control (15) and the drift term is bounded as $r \to \infty$, it suffices to show that

$$P(X^*(t) = 0 \quad \text{for some} \quad t \in [0,T]) = 0 \tag{19}$$

where $(X^*(t),\, t \in [0,T])$ is the (local) solution of the stochastic differential equation (9) with the control (15).

It is known (e.g., [6]) that the so-called two-dimensional Bessel process with the infinitesimal generator

$$\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)$$

does not hit $r = 0$ at a positive time. This process can be defined as the unique strong solution of the scalar stochastic differential equation

$$dZ(t) = \frac{1}{2Z(t)} dt + dB(t) \tag{20}$$

where $(B(t),\, t \geq 0)$ is a standard Brownian motion. There is a $\delta > 0$ such that for $r \in [0,\delta]$ and $t \in [0,T]$

$$\frac{1}{2r} < \coth r - \frac{1}{2} g(t) \tanh r \tag{21}$$

By localization of $(X^*(t),\, t \in [0,T])$ to $(0,\delta]$ and comparison of the stochastic differential equations (9) and (20), it follows by a comparison theorem (p. 352 [5]) that (19) is satisfied. This completes the proof.

A stochastic control problem in real hyperbolic n-space, $\mathbb{H}^n(\mathbb{R})$, for $n > 3$ can be formulated and solved in analogy to the problem here in $\mathbb{H}^3(\mathbb{R})$.

The optimal control is quite natural because if y is given in geodesic polar coordinates then the distance of $\exp y$ from the origin in $\mathbb{H}^3(\mathbb{R})$ is $\tanh \frac{|y|_0}{2}$.

References

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