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Theorems About an Approximate Solution to an Abstract Nonlocal Cauchy Problem

Ludwik Byszewski*
Department of Applied Mathematics
Florida Institute of Technology
Melbourne, FL 32901

Abstract

The aim of the paper is to prove a theorem about the existence of an approximate solution to an abstract nonlinear nonlocal Cauchy problem in a Banach space. The right-hand side of the nonlocal condition belongs to a locally closed subset of a Banach space. The paper is a continuation of papers [1], [2] and generalizes some results from [3].

1. Introduction

In papers [1] and [2] theorems about the existence and uniqueness of solutions of abstract nonlinear nonlocal Cauchy problems in Banach spaces were considered. To obtain those results the Banach theorem about the fixed point and the method of semigroups were used. The aim of this paper is to construct an approximate solution to an abstract nonlinear nonlocal Cauchy problem in a Banach space under the assumptions that the right-hand side of the differential equation does not satisfy any kind of the Lipschitz condition and under the assumption that the right-hand side of the nonlocal condition belongs to a locally closed subset of a Banach space. To prove the main result of the paper a modification of a method used by Lakshmikantham and Leela (see [3], Section 2.6) is applied. To apply so the modified method a special kind of a locally closed subset of a Banach space is established in the paper. The paper, analogously as in [1] and [2], can be applied in physics.

*Permanent address: Institute of Mathematics
Cracow University of Technology
Warszawska 24, 31-155 Cracow, Poland

2. Preliminaries

Let E be a Banach space with norm $\|\ \cdot\ \|$ and let

$$B(a,\rho) := \{x \in E \colon \|x-a\| \le \rho\},$$

where $a \in E$ and $\rho > 0$.

To find an approximate solution for the Cauchy nonlocal problem considered in the paper we shall need the following:

Assumption (A_1). F is such a subset of E that for each $x_o \in F$ there exist numbers $r \in (0,\infty)$ and $\varepsilon \in (0,r)$, and there exists a sequence $\{x_o^i\}_{i=1}^{\infty} \subset F_o \setminus \{x_o\}$, where

$$F_o := F \cap B(x_o, r),$$

such that

(i) F_o if closed in E; (1)

(ii) $\|x_o^i\| \le \|x_o^{i+1}\| \le \|x_o\|$ for all $i = 1,2,\dots$; (2)

(iii) $\|x_o^i - x_o\| \underset{i\to\infty}{\longrightarrow} 0$; (3)

(iv) $\|x_o^i - x_o\| < \varepsilon \qquad$ for all $i = 1,2,\dots$ (4)

It is easy to see that a subset F of a Banach space E satisfying Assumption (A_1) must be a locally closed set.

Now, we shall give two examples.

Example 1. Let $E = \mathbb{R}^2$ with the Euclidean norm and let $F = \mathbb{R} \times (0,c]$, where c is a positive real number. Choose an arbitrary point $x_o = (x_{01}, x_{02})$ from F and choose a number r satisfying the condition $0 < r < x_{02}$. Next, choose a number ε such that $0 < \varepsilon < r < x_{02}$ and define a sequence $\{x_o^i\}_{i=1}^{\infty}$, where $x_o^i = (x_{01}^i, x_{02}^i)$ $(i = 1,2,\dots)$, by the formulae

$$x_{01}^i := x_{01} \quad \text{and} \quad x_{02}^i := x_{02} - \frac{\varepsilon}{i+1}\ (i = 1,2,\dots). \tag{5}$$

Since

$$0 < x_{02} - \frac{\varepsilon}{i+1} < x_{02} - \frac{\varepsilon}{i+2} < x_{02} \quad (i = 1,2,\dots),$$

$$\|x_o^i - x_o\| = \frac{\varepsilon}{i+1} \quad (i = 1,2,\dots)$$

and

$$\|x_o^i - x_o\| \le \frac{\varepsilon}{2} < \varepsilon \quad (i = 1,2,\dots),$$

then the sequence $\{x_o^i\}_{i=1}^{\infty}$ given by (5) satisfies conditions (2), (3) and (4). Additionally, the set $(\mathbb{R} \times (0,c]) \cap B(x_o, r)$ is closed in E. Consequently, sets E and F from this example satisfy Assumption (A_1) and, therefore, there exists a nonempty class of subsets F of a Banach space E such that Assumption (A_1) is satisfied.

Example 2. Let $E = \mathbb{R}^2$ with the Euclidean norm and let $F = (-\infty, 0] \times (-\infty, 0]$. It is easy to see that for each $x_o \in F$ there exists a $r > 0$ such that condition (1) from Assumption (A_1) holds, but for $x_o = (0,0)$ there is not a sequence $\{x_o^i\}_{i=1}^{\infty} \subset F_o \setminus \{x_o\}$ such that conditions (2)–(4) from Assumption (A_1) hold simultaneously. Consequently, there exists a locally closed subset of E such that conditions (2)–(4) do not hold for this subset. Therefore, to find an approximate solution for the nonlocal problem considered in the paper it will be necessary to use Assumption (A_1) in the next section.

In Section 3, under Assumption (A_1) and under some assumptions concerning a function f and constants t_o, T and k, an approximate solution for the following abstract nonlocal Cauchy problem

$$\begin{aligned} x'(t) = f(t, x(t)), \quad t \in [t_o, t_o + T], \\ x(t_o) + kx(t_o + T) = x_o \in F \end{aligned} \tag{6}$$

is studied.

3. Theorem About Approximate Solution

Theorem 1. *Let E be a Banach space with norm $\|\cdot\|$ and let x_o be an arbitrary fixed element of a subset F of space E satisfying Assumption (A_1). Assume, additionally, that*

(A_2) *k is a constant satisfying the condition*

$$0 < |k| < \frac{r - \varepsilon}{r - \varepsilon + \|x_o\|}.$$

(A_3) *$f \in C([t_o, t_o + T_0] \times F, E)$, where t_o is a real constant and T_o is a real positive constant.*

(A_4) *$\|f(t,x)\| \le M - 1$ for $(t,x) \in [t_o, t_o + T_o] \times F_o$, where M is a constant satisfying the inequality $M > 1$.*

(A_5) $T := \min\left\{T_0, \frac{r-\varepsilon}{M}(1 - |k|) - \frac{\|x_o\|}{M}|k|\right\}$.

(A_6) $\liminf\limits_{h \to 0+} \frac{1}{h} d(x + hf(t,x), F) = 0$ *for* $(t,x) \in [t_o, t_o + T_o] \times F$.

(A_7) $F_* = \Big\{a \in F_o \colon \frac{x_o^i - a}{k} \in F \quad (i = 1, 2, \dots),\ \liminf\limits_{h \to O^+} \frac{1}{h}\left\| a + hf(t_o, a) - \frac{x_o^1 - a}{k}\right\| = 0$ *and* $\liminf\limits_{h \to O^+} \frac{1}{h}\left\| \frac{x_o^{i-1} - a}{k} + hf\left(t, \frac{x_o^{i-1} - a}{k}\right) - \frac{x_o^i - a}{k}\right\| = 0$ *for* $t \in (t_o, t_o + T] \quad (i = 2, 3, \dots)\Big\} \neq \emptyset$.

(A$_8$) *$\{\varepsilon_n\}_{n-1}^{\infty}$ is a sequence of numbers belonging to the interval $(0,1)$ and satisfying the condition $\lim_{n\to\infty} \varepsilon_n = 0$.*

Then, for each natural n, problem (6) has ε_n-approximate solution $x_n(t)$ on $[t_o, t_o+T]$ into $B(x_o, r)$ such that the following conditions hold:

There is a sequence $\{t_i^n\}_{i=0}^{\infty}$ in $[t_o, t_o+T]$ such that

(i) *$t_o^n = t_o$, $t_i^n - t_{i-1}^n \le \varepsilon_n$ $(i=1,2,\dots)$ and $\lim_{i\to\infty} t_i^n = t_o + T$;*

(ii) *$x_n(t_o) \in F_*$, $x_n(t_o) + kx_n(t_i^n) = x_o^i \in F_o \setminus \{x_o\}$ $\quad (i=1,2,\dots)$, $x_n(t_o) + kx_n(t_o+T) = x_o \in F_o$ and $\|x_n(t) - x_n(s)\| \le M|t-s|$ for $t, s \in [t_o, t_o+T]$;*

(iii) *$x_n(t_i^n) \in F_o$ and $x_n(t)$ is linear on $[t_{i-1}^n, t_i^n]$ $\quad (i=1,2,\dots)$;*

(iv) *If $t \in (t_{i-1}^n, t_i^n)$, then $\|x_n'(t) - f(t_{i-1}^n, x_{i-1}^n)\| \le \varepsilon_n$ $\quad (i=1,2,\dots)$;*

(v) *If $(t,y) \in [t_{i-1}^n, t_i^n] \times F_o$, with $\|y - x_n(t_{i-1}^n)\| \le M(t_i^n - t_{i-1}^n)$, then $\|f(t,y) - f(t_{i-1}^n, x_n(t_{i-1}^n))\| \le \varepsilon_n$ $\quad (i=1,2,\dots)$.*

Proof. Let n be an arbitrary fixed natural number. We shall construct sequences $x_n(t)$ and $\{t_i^n\}_{i=0}^{\infty}$ by induction on i.

First of all we shall construct ε_n-approximate solution $x_n(t)$ on $[t_o, t_1^n]$. For this purpose let

$$t_o^n := t_o \tag{7}$$

and let $x_n(t_o)$ be an arbitrary chosen fixed element of F_*, i.e.,

$$x_n(t_o) \in F_o \quad \text{and} \quad \frac{x_o^i - x_n(t_o)}{k} \in F \quad (i=1,2,\dots), \tag{8^1}$$

$$\liminf_{h\to O^+} \frac{1}{h} \left\| x_n(t_o) + hf(t_o, x_n(t_o)) - \frac{x_o^1 - x_n(t_o)}{k} \right\| = 0 \tag{8^2}$$

and

$$\liminf_{h\to 0^+} \frac{1}{h} \left\| \frac{x_o^{i-1} - x_n(t_o)}{k} + hf\left(t, \frac{x_o^{i-1} - x_n(t_o)}{k}\right) - \frac{x_o^i - x_n(t_o)}{k} \right\| = 0$$
$$\text{for} \quad t \in (t_o, t_o+T] \quad (i=2,3,\dots). \tag{8^3}$$

The choice of $x_n(t_o)$ is possible according to Assumption (A$_7$).

Next, choose

$$\delta_1^n \in [0, \varepsilon_n] \tag{9}$$

such that δ_1^n is the largest number that the following conditions hold:

(a$_1$) $\delta_1^n \le T$;

(b$_1$) If $t \in [t_o, t_o + \delta_1^n]$ and $y \in F_o$ with $\|y - x_n(t_o)\| \le M\delta_1^n$ then $\|f(t,y) - f(t_o, x_n(t_o))\| \le \varepsilon_n$;

(c$_1$) $d(x_n(t_o) + \delta_1^n f(t_o, x_n(t_o)), F) \le \frac{\varepsilon_n}{2}\delta_1^n$

and

(d$_1$) $\left\| x_n(t_o) + \delta_1^n f(t_o, x_n(t_o)) - \frac{x_o^1 - x_n(t_o)}{k} \right\| \le \varepsilon_n \delta_1^n$.

The above choice is possible, by the fact that $f \in C([t_o, t_o + T_0] \times F, E)$, according to Assumption (A$_6$) and by (8^2).

Now, define

$$t_1^n := t_o + \delta_1^n \tag{10}$$

and

$$x_n(t_1^n) := \frac{x_o^1 - x_n(t_o)}{k}. \tag{11}$$

Since $\delta_1^n > 0$ then, from (10), $t_1^n > t_o$ and, consequently, by (7), (9) and (10), condition (i) holds for $i = 1$.

Moreover, by (11) and (8^1),

$$x_n(t_1^n) \in F. \tag{12}$$

Additionally, from (d$_1$) and from (10)–(12),

$$\|x_n(t_o) + (t_1^n - t_o) f(t_o, x_n(t_o)) - x_n(t_1^n)\| \le \varepsilon_n (t_1^n - t_o). \tag{13}$$

Next, define

$$x_n(t) = \frac{x_n(t_1^n) - x_n(t_o)}{t_1^n - t_o}(t - t_o) + x_n(t_0) \quad \text{for} \quad t \in [t_o, t_1^n]. \tag{14}$$

If $t, s \in [t_o, t_1^n]$ then by (14) and (13), by the assumption that $\{\varepsilon_n\} \subset (0, 1)$, and by Assumption (A$_4$),

$$\begin{aligned} \|x_n(t) - x_n(s)\| &\le \frac{\|x_n(t_1^n) - x_n(t_o)\|}{t_1^n - t_o} \; |t - s| \\ &\le [\|f(t_o, x_n(t_o))\| + \varepsilon_n] \; |t - s| \\ &\le [\|f(t_o, x_n(t_o))\| + 1] \; |t - s| \\ &\le M|t - s|, \end{aligned} \tag{15}$$

which shows that $x_n(t)$ satisfies the Lipschitz condition on $[t_o, t_1^n]$. This, together with (8^1), (11) and Assumption (A$_1$), means that condition (ii) holds for $i = 1$.

Now, we shall show that $x_n(t_1^n) \in F_o$. For this purpose observe that, from (11), (4), (15), (10) and (a$_1$),

$$\begin{aligned} \|x_n(t_1^n) - x_o\| &\le \|x_n(t_1^n) - x_o^1\| + \|x_o - x_o^1\| \\ &\le \|x_n(t_1^n) - x_n(t_o) - k x_n(t_1^n)\| + \varepsilon \\ &\le M(t_1^n - t_o) + |k| \, \|x_n(t_1^n)\| + \varepsilon \\ &\le MT + |k| \, \|x_n(t_1^n)\| + \varepsilon. \end{aligned} \tag{16}$$

Simultaneously, by (11),

$$\|x_n(t_o)\| \le \|x_o^1\| + |k|\,\|x_n(t_1^n)\| \tag{17}$$

and, by (15), (10) and (a$_1$),

$$\|x_n(t_1^n) - x_n(t_o)\| \le MT.$$

Consequently,

$$\|x_n(t_1^n)\| \le MT + \|x_n(t_o)\|. \tag{18}$$

Therefore, from (18) and (17),

$$\|x_n(t_1^n)\| \le MT + \|x_o^1\| + |k|\,\|x_n(t_1^n)\|.$$

Hence, by Assumption (A$_2$),

$$\|x_n(t_1^n)\| \le \frac{MT + \|x_o^1\|}{1 - |k|}. \tag{19}$$

Then, from (16), (19), (2) and from Assumption (A$_5$),

$$\begin{aligned}
\|x_n(t_1^n) - x_o\| &\le MT + |k|\frac{MT + \|x_o^1\|}{1-|k|} + \varepsilon \\
&= \frac{M}{1-|k|}\,T + \frac{|k|}{1-|k|}\,\|x_o^1\| + \varepsilon \\
&\le \frac{M}{1-|k|}\,T + \frac{|k|}{1-|k|}\,\|x_o\| + \varepsilon \\
&\le \frac{M}{1-|k|}\left[\frac{r-\varepsilon}{M}(1-|k|) - \frac{\|x_o\|}{M}\,|k|\right] + \frac{|k|}{1-|k|}\,\|x_o\| + \varepsilon \\
&= r - \varepsilon - \frac{|k|}{1-|k|}\,\|x_o\| + \frac{|k|}{1-|k|}\,\|x_o\| + \varepsilon = r.
\end{aligned} \tag{20}$$

Consequenty, by (12), (20) and by the definition of F_o, $x_n(t_1^n) \in F_o$. This, together with (14), means that (iii) holds for $i = 1$.

If $t \in (t_o, t_1^n)$, then $x'(t)$ exists and hence, from (14) and (13),

$$\|f(t_o, x_n(t_o)) - x_n'(t)\| \le \varepsilon_n.$$

Hence condition (iv) holds for $t \in (t_o, t_1^n)$.

Finally, if $i = 1$ then condition (v) is a consequence of condition (b$_1$).

Assume now that i is a fixed natural number belonging to $\mathbb{N} \setminus \{1\}$, $x_n(t)$ is defined on $\left[t_o, t_{i-1}^n\right]$, where $t_{i-1}^n \le t_o + T$, and conditions (i)–(v) of the thesis of Theorem 1 hold on $\left[t_o, t_{i-1}^n\right]$. Analogously as in the proof of

Theorem 1 for $i = 1$, choose $\delta_i^n \in [0, \varepsilon_n]$ such that δ_i^n is the largest number satisfying the conditions:

(a$_i$) $t_{i-1}^n + \delta_i^n \le t_o + T$;

(b$_i$) If $t \in [t_{i-1}^n, t_{i-1}^n + \delta_i^n]$ and $y \in F_o$ with $\|y - x_n(t_{i-1}^n)\| \le M\delta_i^n$, then $\|f(t,y) - f(t_{i-1}^n, x_n(t_{i-1}^n))\| \le \varepsilon_n$;

(c$_i$) $d(x_n(t_{i-1}^n) + \delta_i^n f(t_{i-1}^n, x_n(t_{i-1}^n)), F) \le \frac{\varepsilon_n}{2}\delta_i^n$

and

(d$_i$) $\|x_n(t_{i-1}^n) + \delta_i^n f(t_{i-1}^n, x_n(t_{i-1}^n)) - \frac{x_o^i - x_n(t_o)}{k}\| \le \varepsilon_n \delta_i^n$.

Since $\delta_i^n > 0$ and $\frac{x_o^i - x_n(t_o)}{k} \in F$, then let

$$t_i^n := t_{i-1}^n + \delta_i^n, \tag{21}$$

$$x_n(t_i^n) := \frac{x_o^i - x_n(t_o)}{k} \tag{22}$$

and

$$x_n(t) := \frac{x_n(t_i^n) - x_n(t_{i-1}^n)}{t_i^n - t_{i-1}^n}(t - t_{i-1}^n) + x_n(t_{i-1}^n) \quad \text{for} \quad t \in [t_{i-1}^n, t_i^n]. \tag{23}$$

Using a similar argument to the argument in the first part of the proof, we obtain properties (i)–(v) of the thesis of Theorem 1 for $t \in [t_o, t_i^n]$. Particularly, if $t, s \in [t_{i-1}^n, t_i^n]$, then by (23), (22), (21), (d$_i$), by the assumption that $\{\varepsilon_n\} \subset (0,1)$, and by Assumption (A$_4$),

$$\begin{aligned} \|x_n(t) - x_n(s)\| &\le \frac{\|x_n(t_i^n) - x_n(t_{i-1}^n)\|}{t_i^n - t_{i-1}^n}|t-s| \\ &\le \left[\|f(t_{i-1}^n, x_n(t_{i-1}^n))\| + \varepsilon_n\right]|t-s| \\ &\le \left[\|f(t_{i-1}^n, x_n(t_{i-1}^n))\| + 1\right]|t-s| \\ &\le M|t-s|, \end{aligned} \tag{24}$$

which shows that $x_n(t)$ satisfies the Lipschitz condition n $[t_{i-1}^n, t_i^n]$. Therefore, to prove the Lipschitz condition on $[t_o, t_i^n]$, it is enough to prove this condition for

$$t < s, \qquad t \in [t_o, t_{i-1}^n], \qquad s \in (t_{i-1}^n, t_i^n]. \tag{25}$$

Since

$$\begin{aligned} \|x_n(t) - x_n(s)\| &\le \|x_n(t) - x_n(t_{i-1}^n)\| + \|x_n(t_{i-1}^n) - x_n(s)\| \\ &\le M(t_{i-1}^n - t) + M(s - t_{i-1}^n) = M(s-t) = M|t-s| \end{aligned}$$

for t, s satisfying (25), then $x_n(t)$ satisfies the Lipschitz condition on $[t_o, t_i^n]$.

To show that $x_n(t_i^n) \in F_o$, observe that, from (22) and (4), from (24) for $t, s \in [t_o, t_i^n]$, and from (21) and (a_i),

$$\|x_n(t_i^n) - x_o\| \leq MT + |k| \, \|x_n(t_i^n)\| + \varepsilon. \tag{26}$$

But, by (22),

$$\|x_n(t_o)\| \leq \|x_o^i\| + |k| \, \|x_n(t_i^n)\|. \tag{27}$$

Simultaneously, from (24) for $t, s \in [t_o, t_i^n]$, (21) and (a_i),

$$\|x_n(t_i^n)\| \leq MT + \|x_n(t_o)\|. \tag{28}$$

Therefore, by (28) and (27),

$$\|x_n(t_i^n)\| \leq MT + \|x_o^i\| + |k| \, \|x_n(t_i^n)\|.$$

Hence, from Assumption (A_2),

$$\|x_n(t_i^n)\| \leq \frac{MT + \|x_o^i\|}{1 - |k|}. \tag{29}$$

Then, by (26), (29), (2) and Assumption (A_5),

$$\begin{aligned} \|x_n(t_i^n) - x_o\| &\leq MT + |k| \, \frac{MT + \|x_o^i\|}{1 - |k|} + \varepsilon \\ &\leq \frac{M}{1 - |k|} T + \frac{|k|}{1 - |k|} \|x_o\| + \varepsilon \leq r. \end{aligned} \tag{30}$$

Consequently, from the fact that $x_n(t_i^n) \in F$, from (30) and from the definition of F_o, $x_n(t_i^n) \in F_o$.

Arguing as in [3] (see [3], Section 2.6), we have that

$$\lim_{i \to \infty} t_i^n = t_o + T$$

and then

$$x(t_o + T) = \lim_{i \to \infty} x_n(t_i^n).$$

Therefore, there is an ε_n-approximate solution $x_n(t)$ on $[t_o, t_o + T]$ into $B(x_o, r)$ such that conditions (i)–(v) from the thesis of Theorem 1 hold.

Theorem 2. *Suppose that the assumptions of Theorem 1 hold and that*

$$\lim_{n \to \infty} x_n(t) = x(t) \quad \textit{for} \quad t \in [t_o, t_o + T].$$

Then $x(t)$ is a solution of problem (6) for $t \in [t_o, t_o + T]$.

Proof. Since the sequence $\{x_n(t)\}$ is equicontinuous for $t \in [t_o, t_o + T]$, by (ii) of the thesis of Theorem 1, it follows that $\{x_n(t)\}$ converges uniformly to $x(t)$ for $t \in [t_o, t_o + T]$ and that $x(t)$ is continuous for $t \in [t_o, t_o + T]$. Moreover, from thesis (i) and (ii) of Theorem 1,

$$x(t_o) + kx(t_o + T) = x_o,$$

and using similar argument as in [3] (see [3], the proof of Lemma 2.6.1), we obtain that

$$x(t) = x(t_o) + \int_{t_o}^{t} f(s, x(s))ds \quad \text{for} \quad t \in [t_o, t_o + T].$$

This completes the proof of Theorem 2.

Last of all, we shall give the following:

Example 3. Let

$$E := \mathbb{R}^2, \qquad F := \mathbb{R} \times (-\infty, 0]$$

and let $x_o = (x_{01}, x_{02})$ be an arbitrary point belonging to F such that $x_o \neq (0,0)$. Moreover, let $x_o^1 = (0,0)$ and let $x_o^i = (x_{01}^i, x_{02}^i)$ $(i = 2, 3, \ldots)$ be an arbitrary sequence belonging to the segment $[x_o^1, x_o]$ and satisfying the conditions

$$\|x_o^i\| \leq \|x_o^{i+1}\| \leq \|x_o\| \quad \text{for all} \quad i = 2, 3, \ldots$$

and

$$\|x_o^i - x_o\| \underset{i \to \infty}{\longrightarrow} 0.$$

Choose two numbers r and ε such that

$$r > \varepsilon > \|x_o^1 - x_o\| = \|x_o\|.$$

Then

$$\|x_o^i - x_o\| \leq \|x_o^1 - x_o\| < \varepsilon \quad \text{for all} \quad i = 2, 3, \ldots$$

and, consequently, Assumption (A_1) holds.

Let k be a real constant satisfying the condition

$$0 < |k| < \frac{r - \varepsilon}{r - \varepsilon + \|x_o\|},$$

t_o be a real constant, T_o be a positive constant and M be a constant such that $M > 1$.

Introduce an arbitrary function f belonging to $C\left([t_o, t_o + T_o] \times F, E\right)$ and satisfying the following conditions:

(i) $\liminf\limits_{h \to 0^+} \frac{1}{h} d(x + hf(t,x), F) = 0$ for $(t,x) \in [t_o, t_o + T_o] \times F$,

(ii) $$f(t_o, hb) := -b - \frac{b}{k} \tag{31}$$

and

$$f\left(t, \frac{x_o^{i-1} - hb}{k}\right) := -\frac{x_o^{i-1} - x_o^i}{hk} \ (i = 2, 3, \dots), \tag{32}$$

where $t \in (t_o, t_o + T]$, $T := \min\left\{T_o, \frac{r-\varepsilon}{M}\,(1 - |k|) - \frac{\|x_o\|}{M}\,|k|\right\}$, $h > 0$ and b is an element of F such that

$$\|b\| \le \frac{|k|}{k+1}\,(M-1),\ hb \in F_o \text{ and } \frac{x_o^{i-1} - hb}{k} \in F \setminus F_o \ (i = 2, 3, \dots),$$

(iii) $\|f(t,x)\| \le M - 1$ for $(t,x) \in ([t_o, t_o + T] \times F_o) \setminus \{(t_o, hb)\}$.

Let $a := hb$. Then, from the above considerations:

(a) $a \in F_o$,

(b) $\dfrac{x_o^i - a}{k} \in F \setminus F_o \quad (i = 1, 2, \dots)$,

(c) $\|f(t_o, a)\| = \left\|-b - \frac{b}{k}\right\| = \frac{k+1}{|k|}\,\|b\| \le M - 1$,

(d) $\left\|a + hf(t_o, a) - \frac{x_o^1 - a}{k}\right\| = \left\|hb + h\left(-b - \frac{b}{k}\right) + \frac{hb}{k}\right\| = 0$,

(e)
$$\begin{aligned}
&\left\|\frac{x_o^{i-1} - a}{k} + hf\left(t, \frac{x_o^{i-1} - a}{k}\right) - \frac{x_o^i - a}{k}\right\| \\
&= \left\|\frac{x_o^{i-1} - hb}{k} + h\left(-\frac{x_o^{i-1} - x_o^i}{hk}\right) - \frac{x_o^i - hb}{k}\right\| \\
&= \left\|\frac{x_o^{i-1}}{k} - \frac{x_o^{i-1}}{k} + \frac{x_o^i}{k} - \frac{x_o^i}{k}\right\| = 0.
\end{aligned}$$

Consequently, Assumptions (A_1)–(A_7) are satisfied. Particularly, function f defined by formulae (31) and (32) satisfies Assumption (A_7).

References

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