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Alexandre Grothendieck's EGA V

Part III

(Interpretation and Rendition of his 'prenotes')

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§8 Dimension of the set of exceptional hyperplanes

8.1. In the previous sections and notably Sections 2 and 3, we have given statements asserting that the set of $\xi \in \check{P}$ such that Y has a certain property P is constructible and that it contains the generic point η or else that the set Z_P of $\xi \in \check{P}$ "exceptional for P" is constructible and is rare, i.e. that its closure is of codimension ≥ 1. (Nota Bene: we suppose that $S = \mathrm{Spec}(k)$).

In certain cases we can make this statement more precise by giving a better upper bound for this codimension, which is important for certain questions. For example, if we see that this codimension is greater than or equal to two, it follows that a "sufficiently general" straight line D of $\check{P}$ does not intersect Z_P, whence the existence (if k is infinite) of "linear pencils" of hyperplane sections Y_ξ (ξ a geometric point of D) all of which have the property P (see Section No.[1] for examples).

From the *writing up* point of view, since the results of the present No. make more precise some results of the previous sections, the question arises

[1] Section number omitted.

if it is necessary to do this catching up in a separate section (or number) or to give a more precise version gradually as we move along. Redactor decidetur.[2]

8.2. Let Z be the set of $\xi \in \check{P}$ such that $\dim Y_\xi > \dim X - 1$ and let us suppose that for every irreducible component irr X_i of X we have $\dim f(X_i) >$[illegible] then Z is of codimension two in $\check{P}$. This follows from 2.1 and 2.2 (which implies that every irreducible component of Y dominates $\check{P}$) and from the dimension theory for the morphism $Y \to \check{P}$. Starting from this result we may give as a corollary the case where we start with a closed subset Z of X and where we consider the dimension of the inverse images Z_ξ in the Y_ξ ($\xi \in \check{P}$) and we may even take for Z the set of $\xi \in \check{P}$ such that there exists an irreducible component of $T_{k(\xi)}$ whose trace on Y_ξ has a dimension which is too large (NB we always assume that for every irreducible Z_i of Z we have $\dim f(Z_i) > 0$).

Finally the most precise statement in this direction and one that results easily from the first announcement (for X irreducible) and from 2.7 is the following modified statement: F being coherent over X, suppose that for every associated prime cycle T for F we have $\dim f(T) > 0$ then the set of $\xi \in \check{P}$ such that ϕ_ξ is not $F_{k(\xi)}$-regular is (constructible and) of codimension ≥ 2. (The notation for ϕ_ξ is that from No. 5). We can give this as the principal assertion, and announce the previous assertions as corollaries, the proof proceeding via one of the corollaries.

Please note that with the preceding notations if $\xi \in \check{P} - Z$, then for every $y \in Y_\xi$ we have $\operatorname{coprof}_y G_{k(\xi)} = \operatorname{coprof}_y G_\xi$ and consequently if $\operatorname{coprof} F \leq n$ then for $\xi \in \check{P} - Z$ we have coproof of $T_\xi \leq n$ in particular if F is Cohen-Macauley then for $\xi \in P^v - Z$, G_ξ is Cohen-Macauley. Finally if F is (S_k) we have that G_ξ is (S_{k-1}). (Reference O_{IV}).

8.3. We notice that if F is (S_k) for one of the $\xi \in \check{P} - Z$ such that ϕ_ξ is f_k (ξ)-regular and G_ξ has a compliment of codimension ≥ 2 [Illegible] even if $F = \mathcal{O}_X, k = 1$, X being geometrically integral of dimension two where ($k = 2$, X being geometrically integral and geometrically normal of dim 3). It is enough to start from a projective integral surface

$$X \subset P^r$$

over k algebraically closed having a point x where X is not Cohen-Macauley, then for every hyperplane passing through x the corresponding hyperplane section Y_ξ admits x as an associated embedded cycle (respectively, we start from a normal (thus S_2) integral variety $X \subset P^r$ of dimension three having a point $x \in X$ where X is not Cohen-Macauley, then the Y's passing through x are not CM, i.e. they are not (S_2) at x.)

In these examples the set of "exceptional" ξ for the property (S_k) contains the hyperplane of $\check{P}$ defined by $x \in P$ and it is of codimension one

[2] Editor decide.

(and not of codimension $\geq$ two). Compare 8.5 below for a general precise result in this direction.

We point out now the following dimensional result:

Proposition 8.4. *Let T be a closed subset of X and suppose that $codim(T, X) \geq k$. Then for every $\xi \in \check{P}$ we have $codim(T_\xi, Y_\xi) \geq k-1$. Let Z be the set of $\xi \in \check{P}$ such that $codim(T_\xi, Y_\xi) = k-1$ (i.e. $codim(T_\xi, Y_\xi) < k$) then Z is a constructible, nowhere dense [rare Fr] subset of $\check{P}$, i.e. $\bar{Z}$ is of codimension ≥ 1 in $\check{P}$.*

In order for it to be of codimension ≥ 2 it is necessary and sufficient that for every irreducible component T_i of X of codimension equal to k and such that $\dim \overline{f(T_i)} = 0$, there should exist one irreducible component X_j of X such that $codim(T_i, X_j) = k$ and $\dim \overline{f(X_j)} = 0$, i.e. if f is as quasifinite and $k > 0$, that T does not have isolated points x such that $\dim_x X = k$.

The first assertion follows immediately from the following lemma 8.4.1 (a) which is a remorseful afterthought to paragraph 5.

Lemma 8.4.1. *Let X be a locally noetherian prescheme, let L be an invertible module over X, ϕ a section of L, $Y = V(\phi)$, T a closed subset of X. Let us assume that $codim(Y, X) \geq k$.*
Then

a) *$codim(T \cap Y, Y) \geq k-1$.*
b) *In order to have*

$$codim(T \cap Y, Y) = k-1$$

i.e. $codim(T \cap Y, Y) < k$
it is necessary and sufficient that there should exist an irreducible component T_i of T contained in Y, and such that $codim(T_i, X) = k$ and such that for every irreducible comnponent X_j of X containing T_i and such that

$$\dim O_{X_j, T_i} = \dim O_{X, T_i} \quad (= k)$$

we have

$$X_j \not\subset Y.$$

The verification of this lemma is immediate due to the general facts in O_{IV}, Chapter IV about dimension. With the assumptions of 8.4, by 8.4.1 (b) we see which ones are the exceptional hyperplanes H_ξ. If we exclude the set Z_0 of $\xi \in P^v$ such that there is an irreducible component R of T or of X such that $\dim f(R) > 0$ and such that R_ξ is of "dimension too large" (a set which is of codimension two and in what follows it does not count the exceptional H_ξ are those for which there exists a T_i with $\operatorname{codim}(T_i, X) = k$ and $\dim f(T) = 0$, $f(T) \subset H$ [interpr.][3] and such that for *every* irreducible component $X_j \supset T_i$ of X with $\operatorname{codim}(T_i, X_j) = k$

[3] Probably H_ξ.

we have $f(X_j) \not\subset H_\xi$. For a given T_i with $\operatorname{codim}(T_i, X) = k$ if there exists an X_j with $\operatorname{codim}(T_i, X_j) =$ [illegible] and such that $\dim f(X_j) = 0$ then we will have $f(X_j) = f(T_i) \not\subset H_\xi$ and consequently ξ would not be exceptional relative to the T_i. If, on the other hand, for every $X_j \supset T_i$ such that $\operatorname{codim}(T_i, X_j) = k$, we have $\dim f(X_j) > 0$ then for $\xi \in \check{P} - Z_0$, ξ is exceptional relative to T_i if and only if $f(T_i) \not\subset H_\xi$; the set of such ξ is (the trace over of $P - Z_0$ a hyperplane of $\check{P}$. This proves 8.4, and also proves the more precise result that the exceptional set is the union of a set of codimension ≥ 2 and of a *union of hyperplanes* determined in an evident way by the above proof.

(We are afraid that the writeup is quite floppy (or perhaps sloppy) [interpr.] since we have reasoned geometrically all the time without saying so, by taking points over an algebraically closed field. Of course, the condition announced in 8.4 is indeed geometric so that we may suppose k algebraically closed and argue only for k-rational points.) Using 8.4; 5.7.4 and the end of 8.2, we obtain:

Corollary 8.5. *Suppose that for all associated prime cycles R_i of F and suppose that F satisfies (S_k), in order that the (constructible) set of points of $\check{P}$ such that ϕ_ξ is $F_{k(\xi)}$ regular and G_ξ is (S_k) should have a complement of codimension at least two it is necessary and sufficient to have the following: $(\Leftrightarrow)$ for every integer $n \geq 0$ we denote by Z_n the set of $x \in T = \operatorname{supp} F$ such that the coprof$_x$ [illegible] that for every irreducible component Z_{ni} of Z_n with codim$(Z_{ni}, T) = n + k + 1$ and* $\dim f(Z_{ni}) = 0$, *there exists an irreducible component T_j of T containing Z_{ni} such that codim$(Z_{ni}, T_j) = n + k + 1$ and* $\dim f(T_j) = 0$.

When f is quasifinite then for every closed subset R of [illegible] we have $\dim f(R) = \dim R$ so that the criterion takes the following form: there does not exist an isolated point z in any one of the Z_n such that $\dim_z T (= \dim F_z)$ is equal to $n + k + 1$. When F is equidimensional of dimension d this condition is vacuous if $d \leq k$ (and indeed we knew it because in this case the [hypothesis] (S_k) on F is nothing else but the hypothesis Cohen-Macauley), and if $d \geq k+1$ it means that the set $Z_{d-(k+1)}$ of points of T where the co-depth of F is $> d - (k+1)$, this set is of codimension $\geq d$, therefore finite due to the hypothesis (S_k) on F is *empty*, i.e. we have:

$$\begin{aligned} &\operatorname{coproof} F \leq d - (k+1) \\ &\text{i.e. true depth of } F \geq k + 1 \end{aligned}$$

(even though, a priori, we only have true depth of $F \geq k$ as a consequence of the property (S_k) and $k \leq d$). If we no longer assume that F is equidimensional it remains that we may express the desired condition in the following simple way:

8.6. For every closed point $x \in \operatorname{supp} F$ such that $\dim F_x \geq k$, we have $\operatorname{prof} F_x \geq k + 1$. The sufficiency is seen immediately by putting $z = x$. The necessity is seen by noticing that for every ξ such that ϕ_ξ is $X_{k(\xi)}$-regular and $x \in Y_\xi$ we have dim $G_{\xi x} =$ dim $F_x - 1$, $\operatorname{prof} G_{\xi x} = \operatorname{prof} F)_x - 1$ so that x makes the above condition fail we have $\operatorname{prof} G_{\xi x} < k$ but dim $G_{\xi x} \geq k$ which shows that G_ξ does not satisfy condition (S_k) at x; but the set of ξ such that $x \in Y_\xi$ is of codimension 1 (NB: I implicitly assumed that k is algebraically closed, the case to which we are reduced immediately.) The preceding general criterion should be evident in the case 8.6.

We now study the points y of Y that are not smooth for Y_ξ relative to $k(\xi)$. We restrict ourselves to the case where $f\colon X \to P$ is *unramified* (practically, it will be an immersion) and where $X \to S$ is smooth. We do not necessarily assume that S is the spectrum of a field. Since f is unramified the canonical homomorphism $f^*(\Omega^1_{P/S}) \to \Omega^1_{X/S}$ is surjective and its kernel is a locally free module over X which we denote $\check{\mathcal{V}}_{X/P}$; when f is an immersion this is nothing else but the conormal module J/J^2 defined by the ideal $\mathcal{J}$ of X in P and we call it in every case the conormal module.

$$0 \to \check{\mathcal{V}}_{X/P} \to f^*(\Omega^1_{P/X}) \to \Omega^1_{X/S} \to 0 \tag{a}$$

Let us observe that we have also over P an exact canonical sequence (which should appear as an example in paragraph 16 for example)

$$0 \to \Omega^1_{P/S}(1) \to \mathcal{E}_P \to \mathcal{O}_P(1) \to 0 \tag{b}$$

(i.e. $\Omega^1_{P/S}$ is canonically isomorphic to the kernel of the canonical homomorphism) $E_P(-1) \to O_P$ deduced from $E_P \to O_P(1)$, then applying f^*:

$$0 \to f^*(\Omega^1_{P/S}(1) \to \mathcal{E}_X \to \mathcal{O}_X(1) \to 0 \tag{b$'$}$$

which gives an explicit description of $f^*(\Omega^1_{P/S})(1)$ over X and allows therefore to identify $\check{\mathcal{V}}_{X/P}(1)$ with a sub-Module locally a direct factor of E_X or again th dual $\nu_{X/P}(-1)$ is canonically isomorphic to a quotient Module of $\check{\mathcal{E}}_X$. Consequently $P(\mathcal{V}_{X/P}(-1)) = P(\mathcal{V}_{X/P})$ can be canonically embedded into $P(\check{\mathcal{E}}_X) = X \times_S \check{P} = X_{\check{P}}$ as a projective sub-fibration over X therefore as a closed subscheme. The latter is necessarily contained in Y (from the fact that $\Omega_{X/P}(1)$ is contained in the kernel of $\mathcal{E}_X \to \mathcal{O}_X(1)$ [illegible].

The underlying set of this prescheme is nothing else but the set of points of $\mathcal{V} = V(\phi)$ which are *singular zeros* (par. 16)[4] of the section ϕ of $\mathcal{O}_{X \times \check{P}}(1,1)$ relative to the base $\check{P}$, i.e. its points with values in the field k over $\check{P}$ are the points x of $Y_k \subset X_k$ such that ϕ_k vanishes to order at least two at x, i.e. such that Y_k is not smooth of relative dimension $r-1$ over k at x. The announced characterization of singular zeros [illegible, ask AG] the elements of a smooth subscheme $P(\mathcal{V}_{X/P})$ of $X_{\check{P}}$ gives immediately from the following statement which deserves to appear as a preliminary

[4] See part VII of these notes [Interpr.]

proposition if $S = \operatorname{Spec} k$ and if H is a hyperplane of P, then $Y = X \times_{\check{P}} H$ is smooth over k of relative dimension $(d-1)$ at the point $x \in Y(k)$ (i.e. x is a non-singular zero, i.e. geometrically non-singular of the section ϕ of $\mathcal{O}_X(1)$ defined by H) if and only if H does not contain the image by ϕ' of the tangent space to X at x (relative to k) or as we say once more (if $f: X \to P$ is an immersion which allows us to identify X to a subscheme of P) if and only if H is not tangent to X at x. This follows trivially from the Jacobian criterion of smoothness or from the definition of a singular zero, once we make precise the sense of the statement, that is to say, that we explained how a vector subspace of the tangent space to P at a point $a (= f(x))$ defines a linear subspace of P (in such a way that it makes sense to say that H does not contain the said vector subspace): of course this comes from the exact sequence (b) above which allows to define a one-to-one correspondence between the subspaces of the tangent space at a and the linear subspaces of P containing a. This correspondence anyway reduces to associating to a linear subvariety passing through a, with its tangent space at a considered as a subspace of the tangent space to P at a.

Such *sorites* grouped together with various *sorites* about linear subvarieties and about grassmanians ought to be given in one or two preliminary paragraphs of course announcing them over arbitrary base. In fact we can do better knowing that the prescheme Y^{sing} of singular zeros of ϕ relative to $\check{P}$ defined in par.16 is nothing else but $P(\check{\mathcal{V}}_{X/P})$ and (since the latter is smooth over S of relative dimension $d + (r - d - 1) = r - 1$ (r being the relative dimension of $\check{P}$ over S) we are under the suitable conditions studied in No. 16 or paragraph 16. In order to verify it, let us notice that by definition Y^{sing} is nothing else but the sub-prescheme of Y of zeros of the section $\Psi = d\phi|Y$ of $\Omega'_{X_{\check{P}/\check{P}}}(1,1) \otimes \mathcal{O}_Y = \Omega^1_{X/S} \otimes \mathcal{O}_Y(1,1)$ [illegible].

We shall give another interpretation of this section from which the conclusion follows immediately. In order to do this let us consider the following diagram of exact sequences over $X_{\check{P}}$ or more generally over any prescheme Z over $X_{\check{P}}$.

Diagram:

$$\begin{tikzcd}
 & 0 & & 0 & & \\
 & \Omega^1_{X/Y} \otimes \mathcal{O}_Z(1,0) \arrow[u] & & G_{\check{P}/S} \otimes \mathcal{O}_Z(0,-1) \arrow[u] & & \\
0 \arrow[r] & \Omega^1_{P/S} \otimes \mathcal{O}_Z(1,0) \arrow[u] \arrow[rr] & & E \otimes \mathcal{O}_Z \arrow[u] \arrow[rr] & & \mathcal{O}_Z(1,0) \arrow[r] & 0 \\
 & \check{\mathcal{V}}_{X/P} \otimes \mathcal{O}_Z(1,0) \arrow[u] & & \mathcal{O}_Z(0,-1) \arrow[u] \arrow[ull, dashed, "\beta"'] \arrow[urr, dashed, "\alpha"] & & \\
 & 0 \arrow[u] & & 0 \arrow[u] & &
\end{tikzcd}$$

[Note: G or $\mathcal{Y}$] where the first column is deduced from (a) by tensoring with $\mathcal{O}_Z(1,0)$ the row is deduced from (b) by tensoring with $\mathcal{O}_Z$ and the column two is deduced from its transpose from the analogous sequence (b^v) relative to $\check{P}$ (obtained by replacing E by $\check{E}$) and tensoring with $\mathcal{O}_Z$. From the very definition of Y, Z is over Y if and only if the composite morphism α from the diagram is zero, i.e. if we can find a factorization $\beta\colon \mathcal{O}_Z(0,-1) \to \Omega^1_{P/S} \otimes \mathcal{O}_Z(1,0)$. If this is the case we can consider its composition with $\Omega^1_{P/S} \otimes \mathcal{O}_Z(1,0) \to \Omega^1_{X/Y} \otimes \mathcal{O}_Z(1,1)$. I say that this is precisely the section ψ which we have introduced above (the verification ought to be essentially mechanical). It is zero if and only if Z is above $\mathcal{V}(\psi)$ (by the very definition of $\mathcal{V}(\psi)$!) but this means also that β can be factored by $\check{\mathcal{V}}_{X/P} \otimes \mathcal{O}_Z(1,0)$, i.e. that the submodule $\mathcal{O}_Z(0,-1)$ of $E \otimes \mathcal{O}_Z$ is contained in the sub-module $\nu_{X/P} \otimes \mathcal{O}_Z(1,0)$ which evidently signifies also that Z is over the sub-prescheme $P(\mathcal{V}_{X/P}(1))$ of $P(\check{\mathcal{E}}_X)$, achieving the proof that we have announced.

Just before this erudite exercise in syntax (for which I have already had to sweat quite a bit) we could remark that from every set theoretic point of view Y^{sing} is of dimension $r-1$ if $S = \operatorname{Spec} k$, whereas $\check{P}$ is of dimension r so that the image of Y^{sing} in $\check{P}$ is of codimension ≥ 1 which gives again 2.12 (of course the argument is not essentially distinct from the one used in 2.12). We note that most often this set is effectively of codimension one (compare below).

Consequently we cannot in general find "linear pencils" of hyperplane sections all of which are smooth. However we shall see that we can often manage to find the pencils formed by hyperplane sections not having any supersingular point due to the fact that in the most common cases the image of $\mathcal{Y}^{\text{sup sing}}$ in $\check{P}$ is of codimension two.

We shall first of all summarize the essential points of differential nature in the situation studied here:

Theorem 8.7.

(a) *The sub-prescheme $\mathcal{Y}^{\text{sing}}$ (defined in No. or par. 16) in the present situation is nothing else but $P(\mathcal{V}_{X/P})$ considered as a sub-scheme of Y as explained above.*

(b) *The underlying set of the prescheme $\mathcal{Y}^{\text{sup sing}}$ (cf No. or par. 16) is nothing else but the set of ramification points of the morphism of smooth preschemes over S of relative dimension $r-1$ and r (namely) $\mathcal{Y}^{\text{sing}} = P(\mathcal{V}_{X/P}) \to \check{P}$, i.e. in order for the latter morphism to be unramified at the point y (ref to the definition) it is necessary and sufficient that y should be geometrically an ordinary singular point for ϕ_ξ (ξ being the point of $\check{P}$ image of y).*

(c) *Let us assume $S = Spec(k)$ and that $y \in \mathcal{Y}^{\text{sing}} = P(\nu)$ is a k-rational point; let [illegible] and ξ be its projections in $X(k)$ resp. $\check{P}(k)$ and let us consider the linear subvariety H^1 of $\check{P}$ "image" of the tangent map of the closure of its image in $\check{P}$, given the induced reduced structure* *and let us consider the induced morphism $g\colon \mathcal{Y}^{\mathrm{sing}} \to T$ (a dominant morphism of integral preschemes). The conditions (i) and (ii) (bis) are equivalent:*

(i) *The morphism g is generically étale (i.e. étale at least one point or what is the same, unramified at the generic point of $\mathcal{Y}^{\mathrm{sing}}$)*

(i bis) *The field extension L/K defined by g is finite and separable.*

(i ter) *The morphism g is birational, i.e. the extension L/K is the trivial extension.*

(ii) $\mathcal{Y}^{\mathrm{sing}} \neq \mathcal{Y}^{\mathrm{sup\ sing}}$ *(as sets)*

(ii bis) *There exists an $x \in X(\bar{k})$ and a tangent hyperplane H to $X_{\bar{k}}$ at x which is not osculating at x by which we understand precisely that x is not supersingular for the section of $\mathcal{O}_{X_{\bar{k}}}(1)$ that defines $H \ldots$).*

These conditions imply that $\mathcal{Y}^{\mathrm{sup\ sing}} \neq \emptyset$ [Fr. illegible] $\dim \mathcal{Y}^{\mathrm{sup\ sing}} \leq r-2$ so that the image of $\mathcal{Y}^{\mathrm{sup\ sing}}$ in $\check{P}$ has a codimension ≥ 2, and they imply also

(iii) $\dim T = r-1$, *i.e. T is of codimension one in $\check{P}$.*

Proof. The equivalence of (i) with (i bis) is trivial, its equivalence with (ii) is a trivial consequence of 8.7 b), finally the equivalence of (ii) and of (ii bis) is practically the definition of $H^{\mathrm{sup\ sing}}$. Evidently (i ter) $\Rightarrow$ (i), it remains to prove that (i) $\Rightarrow$ (i er). We may obviously suppose that k is algebraically closed and we are reduced to proving (taking into account the hypothesis (i)) that there exists an open set $U \neq \phi$ such that $\xi \in U(k)$ implies that there exists exactly one point y of $\mathcal{Y}^{\mathrm{sing}}(k)$ over ξ. This will follow from 8.7c) which implies more precisely.

Corollary 8.9. *Suppose that condition (i) of 8.8 is satisfied and let U be the open subset of T of all points where T is smooth over k. Then $U \neq \emptyset$, $\mathcal{Y}^{\mathrm{sing}}|U \to U$ is an open immersion a fortiori $\mathcal{Y}^{\mathrm{sing}}|U$ does not contain points of $\mathcal{Y}^{\mathrm{sup\ sing}}$. If X is proper over k, then $g\colon \mathcal{Y}^{\mathrm{sing}} \to T$ is surjective therefore $\mathcal{Y}^{\mathrm{sing}}/U \to U$ is an isomorphism and U is the biggest open set of T having the latter property.*

First of all since g is dominating and generically étale we can find at least one non-empty open subset V of T such that $\mathcal{Y}^{\mathrm{sing}} \mid V \to V$ is étale and surjective which implies that V is smooth over K. If then $\xi \in V(K)$ and if y is a point of $\mathcal{Y}^{\mathrm{sing}}(k)$ over ξ, then with the notations of 8.7 c) the space H' is nothing else but the tangent space to T at ξ, and as we observed here this implies that the point x of $X(k)$, the projection of y is determined as the orthogonal point to H', thus it is uniquely determined thus since $\mathcal{Y}^{\mathrm{sing}} \subset X \times \check{P}$, y is uniquely determined.

This proves already that g is birational (being generically étale and generically radical). On the other hand the morphism ψ (whose definition is evident) which associates to every $\xi \in U(k)$ the unique point $x = \psi(\xi) \in P$ orthogonal to the tangent space to U at ξ, coincides on V with the composition $V \to \mathcal{Y}^{\mathrm{sing}} \bigm| V \to X$, where the second arrow is the projection;

therefore setting $h = (\psi, id): U \to P \times T$, $g_1 = g \mid g^{-1}(U): g^{-1}(U) \to U$ the composition $hg_1: g^{-1}(U) \to P \times T$ is nothing else but the canonical inclusion, this being so for its restriction to $g^{-1}(V) \xrightarrow{\sim} V$. It follows that h factors through the scheme theoretic closure $\overline{\mathcal{Y}^1}$ of $\mathcal{Y}^1$ in $P \times T$ thus that the inverse image of $\mathcal{Y}^1$ (which is open in the above closure) by h is an open subset of U, let us call it U^1. Because of $hg_1 =$ inclusion we see immediately that g^1 induces an isomorphism $g^{-1}(U) \simeq U'$ and this shows that $g^{-1}(U) \to U$ is an open immersion. When X is proper over k, i.e. closed in P, then g is proper and therefore surjective; then $g_1: g^{-1}(U) \to U$ is surjective, thus an isomoprhism. However, if W is an open subset of T such that $g^{-1}(U) \xrightarrow{\sim} W$ is an isomorphism it follows that W is smooth since $\mathcal{Y}^1$ is smooth, thus $W \subset U$. This proves 8.9

The final assertions of 8.8 $\mathcal{Y}^{\text{sup sing}} = \phi$ or $\dim \mathcal{Y}^{\text{sup sing}} = r - 2$ and $\dim T = r - 1$ are trivial: the first one follows from the fact that $\mathcal{Y}^{\text{sing}}$ is irreducible of $\dim r$ and from the fact that $\mathcal{Y}^{\text{sing}}$ or $\mathcal{Y}^{\text{sup sing}}$ [illegible] is defined by the vanishing of a section D of an invertible Module; the second from the fact that L being finite over K we have $\deg \operatorname{tr} L/k = \deg \operatorname{tr} K/k$, i.e. $\dim T = \dim \mathcal{Y}^{\text{sing}} = r - 1$.

Remark 8.10. As we remarked in 8.9 with the notations of the corollary we have $g^{-1}(U) \subset \mathcal{Y}^{\text{sing}} - \mathcal{Y}^{\text{sup sing}}$ but we notice that even if X is closed in P this inclusion is not necessarily an equality, in other words (noting that $g^{-1}(U)$ is nothing else but the set of points where g is étale, so that $\mathcal{Y}^{\text{sing}} - \mathcal{Y}^{\text{sup sing}}$ is the set of points where g is *unramified*) there may be points y of $\mathcal{Y}^{\text{sing}}$ where y is unramified but not étale, (which implies in addition that $g(y)$ is a point that is not geometrically normal and even not geometrically unibranch of T). In geometric terms this corresponds to the following phenomenon; we may have a tangent non-osculating hyperplane for X at a point $x \in X(k)$ such that there exists another point $x^1 \in X(k)$ at which the same hyperplane is tangent at x. Indeed there are obvious examples with P of dim two, X a non-singular curve of degree ≥ 4, in any characteristic. [Note here: the "double tangents" of X correspond to the double points of the "dual curve."]

Corollary 8.11. *Let us assume that k has characteristic zero. Then*

(a) *The image of $\mathcal{Y}^{\text{sup sing}}$ in $\check{P}$ is of codimension ≥ 2.*
(b) *The condition (iii) of 8.8 is equivalent to other conditions, i.e. the negation of the other conditions, as $\mathcal{Y}^{\text{sing}} = \mathcal{Y}^{\text{sup sing}}$ means also that the image of $\mathcal{Y}^{\text{sing}}$ in $\check{P}$ is of codimension ≥ 2.*

Evidently, the assertion (b) implies (a) taking into account 8.8. But by dimension theory, (iii) means that L/K is a finite extension (we could put it in the form ((iii) bis) in 8.8 and in characteristic zero, L is anyway separable over K hence the condition (i bis) of 8.8.

Remark 8.12. Geometrically the assertion (a) means essentially that for a sufficiently general linear pencil of hyperplane sections, every member of the pencil is smooth or has only for geometric points, singular points, ordinary points (and in fact as one sees immediately it can be said in statement (a) consequently in a form a little more precise – we have at most *one* such singular geometic point). The assertion (b) means essentially that if $\mathcal{Y}^{\text{sing}} = \mathcal{Y}^{\text{sup sing}}$ (which can be expressed analytically by the vanishing of a certain section D of an invertible module $\omega_{X/k}^{\otimes 2} \otimes \mathcal{O}_{\mathcal{Y}'}(1,1)$ over $\mathcal{Y}^1$), then for a sufficiently general linear pencil of hyperplane sections all the members of the pencil are smooth. This second situation (whether or not we are in characteristic zero) should entirely be considered as exceptional. The linear variety L in $\ldots T = L$ [illegible]. In classical language it is expressed, if there is no error, by saying that X is ruled for the considered projective immersion [and if we so please] we have here all that we need due to 8.5 and its corollaries to make explicit and justify such a terminology in case if you feel inspired to make connection with this [la taupe, Fr]. For example if $\dim X = 1$ this implies that X is a straight line [illegible] as if $x \in X(k)$ so T contains [illegible]. (b) If the characteristic is $p > 0$, we should normally give examples (with $\dim P = 2$, X a non-singular algebraic curve) where the conditions of 8.6 are not satisfied, i.e. $\mathcal{Y}^{\text{sing}} = \mathcal{Y}^{\text{sup sing}}$ and where nevertheless $\dim T = r - 1$, i.e. examples where L/K is a finite inseprable extension. I am too lazy to construct the examples but I do not doubt that such examples exist.[5]

In (a) in order to take foothold in the following No. where we shall prove that if the exceptional 'ruled' case arises then by a trivial modification of the projective immersion we find ourselves again in the "general" situation of 8.8. N.B. X is said ruled for f if $\operatorname{Im}(\mathcal{Y}^{\text{sing}} \to \check{P})$ is of codimension 2.

The part of the present section from 8.6 to here could (without a doubt) be made into a separate section of a differential character, whereas the beginning of this No. with the one that follows should be merged together into a No. concerning the dimension of exceptional hyperplanes. I only use the fact that $\mathcal{Y}^{\text{sing}}$ has dimension $(r-1)$, where $r = \dim(\check{P})$.

Proposition 8.13. *We always assume that $f\colon X \to P$ is unramified and that X has no isolated points. We assume that X satisfies (R_k) geometrically.*

Let Z_k be the part of $\check{P}$ complement of the set of $\xi \in \check{P}$ such that ϕ_ξ should be $X_{k(\xi)}$-regular and Y_ξ satisfies the geometric condition R_k then:

a) *In order for Z_{k-1} to be of codimension ≥ 2 in $\check{P}$ it suffices that every irreducible component X'_i of X' of dimension $\leq k$ should be ruled for f.*
b) *In order to have Z_k of codimension ≥ 2 in $\check{P}$ it suffices that every irreducible component X_i of X of dimension $\leq k-1$ should be made up of smooth points of X and should be somewhat different.*

[5] Editors Note: An open problem.

Indeed for every ξ the geometrically singular set of Y_ξ (NB: We restrict ourselves to ξ such that ϕ_ξ is $X_{k(\xi)}$-regular which is harmless because of 8.2) and is the union of singular geometric loci sing(Y'_ξ) and of the inverse image T_ξ of T in Y_ξ so that the codimension of this singular set in Y_ξ is equal to the infimum of the codim (sing(Y'_ξ), Y'_ξ) and of the codim (T_ξ, Y_ξ). Let us restict ourselves to ξ such that sing(Y'_ξ) is finite (which is harmless, this leads us to place ourselves in the complement of a set of codimension ≥ 2). The singular geometic points of Y'_ξ are therefore isolated. The conclusion follows easily from this and from 8.4.

Combining 8.13, 8.5 and 8.6, we find in the usual manner

Corollary 8.14. *We suppose that $f: X \to P$ is unramified and that X has no isolated points [illegible] n.*

a) *Suppose X is separable over k. In order that the set of $\xi \in \check{P}$ such that $\phi_\xi$is $X_{k(\xi)}$-regular and Y_ξ is separable, should have a complement of codimension at least two it is necessary and sufficient that every irreducible component X_i of X of dimension one should be formed from smooth points of X and should be ruled relative to f and that for every closed point x of X such that $\dim_x X \geq 2$ we have prof$_x X \geq$ 2, (conditions that are automatically satisfied if X is geometrically normal and if all of its irreducible components are of $\dim \geq 2$).*
b) *Let us assume that X is geometrically normal, in order that the set of $\xi \in \check{P}$ such that ϕ_ξ is $X_{k(\xi)}$-regular and Y_ξ is geometrically normal should have a complement of codimension at least two, it is necessary and sufficient that every irreducible component X_i of X of dimension ≤ 2 should be formed of smooth points of X and that it should be ruled relative to f and that in addition for every closed point x of X such that $\dim_x X \geq 3$ we have prof$_x X \geq 3$.*

Remark 8.15. In 8.6, 8.13, and 8.14 we make for X the hypothesis (S_k) (resp. (R_k) respectively: separable, respectively geometrically normal) that we wish to recover as a conclusion for the hyperplane sections except perhaps for ξ from an exceptional set of codimension at least two.

This does not restrict the generality; to tell the truth, it would have been better to get rid of this preliminary hypothesis, since we see immediately with the help of results of par. 3.4 and 5.12 that if X does not satisfy the hypothesis in question, then (by par. 5) if there exists a closed point x where the hypothesis fails then for every ξ such that ϕ_ξ is $X_{k(\xi)}$-regular condition that only eliminate a set of codimension (illegible) and such that $x \in Y_\xi$ (condition that describes a set of exact codimension one), Y_ξ does not satisfy the said hypothesis at x, the exceptional set $Z \subset \check{P}$ is of codimension one and not two. (I may have somewhat exaggerated the case R_k where we still need some condition, (S_1) and perhaps of equidimensionality . . .)

Remark (remorse)**.** In 8.13 and 8.14 it is enough to suppose that $f\colon X \to P$ is unramified at the smooth points of X; for the sufficiency part it suffices only that they should be unramified over an open subset X' of X where the complement has codimension $\geq 2+1$.

Proposition 8.16. *Let us suppose $f\colon X \to P$ unramified on an open subset complementary to a set of codimension at least two, X geometrically normal and of depth at least three at its closed points, finally X geometrically integral and proper over k. Then the set of $\xi \in \check{P}$ such that Y_ξ is geometrically normal and geometrically integral of dimension equal to* $\dim X - 1$ *(is constructible and) has a complement of codimension at least two.*

Indeed by 8.14 b) such is the case for the property "Y_ξ is geometrically normal of dimension $\dim X-1$" (the dimensional property expresses that ϕ_ξ is $X_{k(\xi)}$-regular.) Therefore, by 6.1 all the Y_ξ are geometrically connected. Since Y_ξ is geometrically normal it is geometrically integral if and only if it is geometrically connected, which gives the proof.

Remarks 8.17.

a) The hypothesis of 8.16 implies that $\dim X \geq 3$. It is possible that wherever X is geometrically irreducible and that $\dim f(X) \geq 3$ (without the hypothesis of normality and of non-ramification) the set of ξ such that Y_ξ is geometrically irreducible has a complement of codimension at least two. We can prove in every case that it does not contain a hyperplane (see below).

b) The conclusion of 8.16 is false if we leave out the assumption that $\mathrm{prof}_x X \geq 3$ for x closed, for example it is false for a non-singular quadric X in P^3; as all tangent hyperplanes are reducible (in fact formed by pairs of concurrent lines) and they form moreover a two dimensional family thus of codimension one in $\check{P}$ (indeed the dual of the quadric is a quadric in the dual space relative to the dual form...). In the case of a non-singular surface in a projective space this situation however should be considered exceptional cf. the following No.

However let us assume that X is smooth of dimension 2, geometrically integral and proper over k and let f be an immersion. Then it follows from 6.1 and 8.8 and 8.14 that if $\mathcal{Y}^{\mathrm{sing}} \to \check{P}$ is not generically finite and inseparable, the set of all $\xi \in \check{P}$ such that Y_ξ is separable over $k(\xi)$ with at most two geometrically irreducible complements is of codim at least two.

We shall now examine more precisely the case of surfaces (the case of curves does not arise evidently, from the point of view of irreducibility of hyperplane sections).

(NB: I noticed with dismay that the quadric is not entitled to be called ruled in the sense that I have been using this word. This is in disagreement with our fathers and it would be necessary to invent a more adequate word for the notion used here.)

Proposition 8.18. *Let us suppose that k is algebraically closed, X is integral (respectively integral and normal) of dimension ≥ 2 and proper over k, let T be a closed finite subset of X such that $X - T$ is smooth and let $f|X - T$ be unramified. In order that the set of $\xi \in \check{P}$ such that Y_ξ should be geometrically irreducible (respectively geometrically integral) of dimension $d-1$ should have a complement of codimension ≥ 2 it is necessary and sufficient that the following conditions should be satisfied:*

a) *For every $x \in T$ there exists a hyperplane section Y_ξ ($\xi \in P(k)$) passing through x of dimension $d-1$ and which is irreducible,*
b) *$X' = X - T$ is "ruled" (sic) for f* **or** *there exists a hyperplane section Y'_ξ ($\xi \in \check{P}(k)$) of X' which is of dimension $d-1$ singular and irreducible [illegible].*

Let us first assume that X is geometrically normal. We have already seen then by 8.14 a) that we can find a closed *subset* Z' of P of codim ≥ 2 such that $\xi \in P - Z'$ implies that Y_ξ is *separable* over $k(\xi)$ and of dimension $d-1$ for such a ξ, it amounts to the same that Y_ξ should be geometrically irreducible or geometrically integral, and the two problems in 8.18 are therefore equivalent. On the other hand, by 5.6, the set U of $\xi \in \check{P}$ such that Y_ξ is geometrically integral of dim $d-1$ (the dimension hypothesis stating that ϕ_ξ is $X_{k(\xi)}$ regular) is *open*. We will exhibit a non-empty open *evident* subset $P - Z$ contained in U and taking for Z the *union of* $\overline{g(\mathcal{Y}'^{\mathrm{sing}})}$ with the hyperplanes $\check{H}_x$ of $\check{P}$ defined by the $f(x)$, $x \in T$. For $\xi \in \check{P} - Z$, Y_ξ is *smooth* of dimension $(d-1)$ and since it is geometrically connected by 6.1 it is geometrically integral. We have, therefore, to say that every irreducible component of codimension one of Z meets the open set U. But these irreducible components are the $\check{H}_x$ [they are repeated possibly, but it is not essential] and also $\overline{g(\mathcal{Y}'^{\mathrm{sing}})}$ when the latter are indeed of codimension one, i.e. X' "not ruled" for f (Nota Bene: we use the irreducibility of $\mathcal{Y}'^{\mathrm{sing}}$. On the other hand, in order that this latter set should meet the open set U it is necessary and sufficient that $(\overline{g\mathcal{Y}'^{\mathrm{sing}}})$ which contains an open and dense set) should meet U. This proves 8.18 in this case. If we do not suppose that X is normal, we apply the previous result to the normalization of X the reasoning is immediate and I do not give the details here. N.B. In the *case* [respective] 8.18 is contained in 8.16 more precisely except in the case $d = 2$. It is for the case [*not respective*] *that it may be better not to require* $d = 2$ but only $d \geq 2 \ldots$.

It remains to make explicit the conditions a) and b) of 8.18. This leads us to examine in a general way the following situation. We suppose that X is geometrically irreducible over k and we (give ourselves) consider a linear subvariety L of $\check{P}$ (corresponding to the question of studying the hyperplane sections of X, passing through a given point x or tangent to X at a given smooth point), formed therefore by the hyperplane containing a linear subvariety L^0 of $\check{P}$ (resp. a point, or the image of a tangent space to X at a smooth point in the two cases considered) and we ask the question if for the generic point ξ of L (therefore for all the points of a non-empty open subset of L) Y_ξ is geometrically irreducible of dim $=$ dim $X-1$. This is a variant of Bertini's theorem, which should appear in No. 3, and is treated by exactly the same method. The dimension question is simply stated for $f(X) \not\subset L^0$ or, if we prefer, get reduced to it. i.e. if $X' = f^{-1}(P - L^0)$ is a dense open subset of X. Let Q be the projective space of hyperplanes passing through L^0 (N.B. if L^0 is defined by a vector subspace F^0 of E we have $Q = P(F^0)$ and we consider the canonical morphism (deduced from $F^0 \to E$, cf. Chap II).

$$u: P - L^0 \to Q$$

and we consider

$$g = uf': f^{-1}(P - L^0) = X' \to Q$$

so that $L \simeq\!\!\to \check{Q}$ and the family of X'_ξ ($\xi \in L$) is nothing else than the family of hyperplane sections relative to the morphism g. On the other hand, we see immediately that for every $\xi \in L$, "*general*" X'_ξ is dense in X_ξ, so that X'_ξ is geometrically irreducible if and only if X is such. This being assumed, the theorem of Bertini-Zariski shows us that we have the wanted conclusion of irreducibility provided that dim $g(X') \geq 2$. (To tell the truth, one could give a converse of 3.1 as follows: If X is geometrically irreducible, Y is geometrically irreducible if and only if either dim $f(X) \neq 2$ *or* dim $f(X) = 1$ and $f(X)$ is contained in a straight line D defined over $\bar{k}$ and the generic fiber of $X \to D$ is geometrically irreducible.) This also allows us in the present version with L to have a necessary and sufficient condition of geometric irreducibility of Y_ξ, ξ generic in L.

From this [illegible] point of view and in terms of field theory we can express the condition in terms of transcendence degree in the following fashion. We choose a "hyperplane at infinity" containing neither L^0 nor X and we place ourselves in its complement, i.e. on a scheme of affine type essentially. We choose a basis of the space of linear forms vanishing on L^0, let it be $T_1, \ldots, T_p$ ($p = \mathrm{codim}(L^0, P)$) and we consider their inverse images $t_1, \ldots, t_p$ in the field of fractions K of X (X assumed integral domain). At least one of the t_i, let us say t_1 is $\neq 0$. Let us consider therefore $a_1 = t_2/t_1, \ldots, a_{p-1} = tp/t_1$ then dim $g(X')$ is nothing else but the transcendence degree of $k(a_1, \ldots\ a_{p-1}) \subset K$ over k. Therefore if the transcendence degree is ≥ 2 we are o.k. If it is one then we must require that over $\bar{k}$, $f(X)$ is contained in a linear subvariety of P containing L^0 and of dimension augmented by one and that the *generic* fiber of $g: X' \to \overline{g(X')}$ should be geometrically irreducible.

Let us suppose that L^0 is of dimenison q, so that the fibers of $u: P - L^0 \to Q$ are of dimension $q+1$ so that those of g are of dim $\leq q+1$, and consequently we have

$$\dim g(X') \geq \dim f(X) - (q+1)$$

so that the dimension condition for $g(X')$ is verified in view of the fact that

$$\dim f(X) \geq q + 3.$$

When $q = 0$ we find the fact indicated in 8.17 a). Returning to conditions of 8.18 we see that condition a) relative to an $x \in T$ is satisfied provided x is not "conical at x relative to f" in an obvious sense. Maybe it will be better to introduce these latest *Bertinian* developments in the next section..., Change of Projective Embedding.