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Some unsolved problems on the prevalence of ergodicity, instability, and algebraic independence

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1. A problem on ergodicity

Let I denote the interval $[0,1]$ and I^n the n-dimensional cube. In their celebrated paper [15] Oxtoby and Ulam have shown that "almost all" measure preserving homeomorphisms of I^n onto itself ($n \geq 2$) are ergodic, i.e., they do not have any measurable invariant subsets in I^n of measure strictly between 0 and 1. Here "almost all" is in the sense of Baire's category. Namely, the set of ergodic homeomorphisms is comeager relative to the metric

$$d(h_1, h_2) = \max\left(\max_{x\in I^n} \|h_1(x) - h_2(x)\|, \max_{x\in I^n} \|h_1^{-1}(x) - h_2^{-1}(x)\|\right).$$

This metric is a complete separable metrization of the composition group of all autohomeomorphisms of I^n; moreover it is such that the subgroup of measure preserving homeomorphisms is topologically closed.

However, the theorem of Oxtoby and Ulam may lack physical significance in the same sense in which the proposition that the set of sequences $(x_0, x_1, \dots) \in I^\omega$ for which the sequence of first means diverges is comeager, seems to lack physical significance. It is the opposite proposition which says that this set is of measure zero (a consequence of the strong law of large numbers) which seems to be physically significant.

Recently Hunt, Sauer and Yorke [4] have introduced a concept which raises the hope that a measure-theoretic version of the Oxtoby–Ulam theorem is possible. Generalizing slightly the definitions of [4] we will describe this as follows.

Recall that every first countable topological group has a left (or right) invariant metrization (see [8]). By a probability measure in a complete metric space we mean a complete Borel probability measure (see [14]).

Let G be a complete metric group. A set $X \subseteq G$ will be called *shy* if there exists a probability measure m over G such that $m(gXh) = 0$ for all $g, h \in G$. (Notice that no invariance of m is required.) Such an m will be called *left and right transversal* to X.

A set $X \subseteq G$ will be called *prevalent* iff the complement of X is shy.

Three fundamental theorems proved in [4] can be easily generalized to the above situation. For convenience of the reader let me state and prove these generalizations. (We differ from [4] by not assuming that G is abelian.)

The first theorem relates the concept of shyness to the standard concept of Haar measure zero.

Theorem 1. *If G is a metric locally compact group, which has a basis of open sets of cardinality less than the first real-valued measurable cardinal and $Y \subseteq G$, then the following three conditions are equivalent:*

(i) Y is shy;

(ii) some (or all) Haar measure of Y vanishes;

(iii) there exists a Borel probability measure on G (no invariance assumed) which is left (or right) transversal to Y.

Proof. Condition (ii) makes sense because all Haar measures are absolutely continuous relative to each other. We must be careful because the Haar measures are not Borel measures in the usual sense (they are defined on the σ-ring generated by compact sets and completed). Nevertheless it is easy to check that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). It remains to show that (iii) $\Rightarrow$ (ii). Let μ be any left Haar measure on G and m be left transversal to Y as in (iii). By Theorem 16.4 of [14], we can assume that m has a compact support, and we define

$$\nu(X) = \int_G m(g^{-1}X)\mu(dg).$$

So $\nu(Y) = 0$ and, in view of the above, it suffices to prove that ν is a Haar measure. In fact ν is left invariant since

$$\begin{aligned}\nu(hX) &= \int_G m(g^{-1}hX)\mu(dg) = \int_G m((h^{-1}g)^{-1}X)\mu(dg) \\ &= \int_G m(k^{-1}X)\mu(hdk) = \nu(X). \qquad \square\end{aligned}$$

The above theorem shows that, for all that matters, there is nothing new about shyness or prevalence in the case of locally compact groups. However, if G is not locally compact these concepts are new and pose some unsolved problems:

(P_0) Does the existence of a Borel probability measure left transversal to Y imply that Y is shy?

Left and right transversality are used in the proof of the following two fundamental theorems (proved in [3] for the abelian case).

Theorem 2. *If Y_1 and Y_2 are shy, then $Y_1 \cup Y_2$ is also shy, i.e., shy sets constitute an ideal.*

Proof. Let μ and ν be the Borel probability measures over G left and right transversal to Y_1 and Y_2 respectively. We define

$$m(X) = \mu \times \nu\{(x, y) \in G^2 : xy \in X\}.$$

We will show that m is left and right transversal to Y_1 and Y_2. We have

$$m(gY_1h) = \int_G \mu(gY_1hy^{-1})\nu(dy) = 0;$$

and

$$m(gY_2h) = \int_G \nu(x^{-1}gY_2h)\mu(dx) = 0.$$

Hence $m(g(Y_1 \cup Y_2)h) = 0$. □

Consider the additive group B of bounded sequences of real numbers with the usual norm $||(x_0, x_1, \dots)|| = \sup\{|x_i| : i = 0, 1, 2, \dots\}$. It is easy to see that each cube $[-n, n]^\omega$ is shy in B. Also $\bigcup_{n=0}^{\infty} [-n, n]^\omega = B$. Hence the assumption that G is *separable* (i.e., second countable) is essential in the next theorem.

Theorem 3. *If G is separable and $Y_1, Y_2, \dots$ are shy, then $Y_1 \cup Y_2 \cup \cdots$ is also shy, i.e., if G is separable shy sets constitute a σ-ideal.*

Proof. Let m_j be a Borel probability measure left and right transversal to Y_j. As well known (see [14]), since G is separable, there exists a compact set C_j with diameter $\leq 1/2^j$ and $m_j(C_j) > 0$. By restricting, normalizing and translating m_j we can assume without loss of generality that $m_j(C_j) = 1$ and that the unity of G belongs to C_j. Notice that if $g_j \in C_j$ for $j = 1, 2, \dots$, then the infinite product $g_1 g_2 \cdots$ in the sense of group multiplication converges (by the assumption about the diameters of the C_j's). Let m^Π be the product measure of the measures m_j in the product space $\prod C_j$. We define

$$m(X) = m^\Pi\{(g_1, g_2, \dots) \in \prod C_j : g_1 g_2 \cdots \in X\}.$$

We will prove that m is left and right transversal to $Y_1 \cup Y_2 \cup \cdots$. We have

$$g_1 g_2 \cdots \in hY_ik \iff g_i \in (g_1 \cdots g_{i-1})^{-1} hY_ik(g_{i+1} g_{i+2} \cdots)^{-1}.$$

Let $\Pi^i C_j = C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times C_{i+2} \times \cdots$ and m_i^{Π} be the product measure of $m_1, \ldots, m_{i-1}, m_{i+1}, m_{i+2}, \ldots$ in $\Pi^i C_j$. Then, since m_i is left and right transversal to Y_i,

$$m(hY_i k) = \int_{\Pi^i C_j} m_i((g_1 \cdots g_{i-1})^{-1} hY_i k(g_{i+1}g_{i+2}\cdots)^{-1}) m_i^{\Pi}(dg) = 0.$$

So m is left and right transversal to each Y_i and hence also to $Y_1 \cup Y_2 i \cup \cdots$. □

Of course Theorem 2 gives us the same feeling about the concept of shyness which we have about the concepts of meagerness or measure zero. But still we have a few basic questions

(P_1) Suppose that $Y \subseteq G$ is shy and that G is metric complete (as always) and separable. Does there exist a shy $\mathbb{G}_\delta$ including Y?

(P_2) Suppose that $X \subseteq G$ is not shy and is *universally measurable*, i.e., measurable relative to any Borel measure in G. Must XX^{-1} contain a neighborhood of unity?

(P_3) Let D be a countable subgroup everywhere dense in G and $X \subseteq G$ a universally measurable set which is left invariant relative to D, i.e., $DX = X$. Must X be shy or prevalent?

Now we state the main problem of this section.

(P_4) Is the set of ergodic homeomorphisms (in the group $\mathbf{H}$ of all measure-preserving autohomeomorphisms of I^n with the distance d defined above) prevalent? (And the same question about the groups of measure preserving diffeomorphisms and Lipschitz homeomorphisms with appropriate complete metrizations.)

(P_5) Does there exist a one-parameter subgroup L of $\mathbf{H}$ such that for every $g, h \in \mathbf{H}$ the sets gLh contain at most countably many non-ergodic homeomorphisms?

Of course (P_5) $\Rightarrow$ (P_4), since the Lebesgue measure in L will be right and left transversal to the set of non-ergodic homeomorphisms.

2. Problems on stability.

In [13] we have represented the states of consciousness of the brain at times $t = 0, 1, 2, \ldots$ (the unit of time represents a small fraction of a second) in the form of vectors $x_t \in I^n$. Here $x_t = (x_{t1}, \ldots, x_{tn})$, where x_{tk} denotes the intensity of firing of the kth neuron (from a certain population of n neurons) at time t. The brain and its memory is represented by a mapping $f : I^n \to I^n$ such that

$$x_{t+1} = f(x_t) \quad \text{for} \quad t = 0, 1, 2, \ldots \tag{1}$$

Thus $x_t = f^t(x_0)$, where f^t is the t-th iterate of f, and $x_0, x_1, \ldots$ represents the train of thoughts (and emotions). We conjectured in [13] that, what is observed as the freedom of choice (free will) in the process of thoughts and decisions, corresponds to the instability of the dynamic system $\langle I^n, f\rangle$, or the "unpredictability" of its trajectories. Let us define f to be *unstable* iff there exists an $\delta > 0$ such that for almost all pairs $(x_0, y_0) \in I^n \times I^n$ we have

$$\sup\{\|f^t(x_0) - f^t(y_0)\| : t = 0, 1, \ldots\} \geq \delta.$$

Now, consider the space of all measurable functions $f : I^n \to I^n$ under some natural metric, e.g.,

$$\int_{I^n} \|f_1(x) - f_2(x)\| dx.$$

(P_6) Are "almost all" functions f unstable? (Here the only natural concept of "almost all" which appears available is: comeager in the sense of Baire's category. However, we may restrict f to measurable and measure preserving bijections of I^n onto itself, or, to autohomeomorphisms of I^n. Conditions of smoothness could be also added. Then we get composition groups and the notion of prevalence defined in Section 1 is available.)

(P_7) Do there exist any measure preserving ergodic autohomeomorphisms of I^n or of the n-dimensional sphere S^n ($n \geq 2$) which are stable? (Of course for the circle S^1 or the torus $(S^1)^n$ the answer is YES.)

If f is continuous then, of course, $x_t = f^t(x_0)$ is a continuous function of x_0. Therefore, given t, if we start to think in sufficiently similar states of mind x_0 and x_0', we will reach similar conclusions $f^t(x_0)$ and $f^t(x_0')$. But, if f is unstable, then this will not persist for large values of t.

Now instead of the equation (1) consider the differential equation

$$\frac{dx}{dt} = f(x), \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Recall the theorem of Picard which tells us that, if f is Lipschitz, i.e.,

$$\|f(x) - f(y)\| \leq C\|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

then for every $x_0 \in \mathbb{R}^n$ there exists a unique solution $x : \mathbb{R} \to \mathbb{R}^n$ of (2) satisfying $x(0) = x_0$. Moreover if we restrict x to the interval $[0, 1]$, then x depends continuously on x_0 in the topology of uniform convergence. Also, for every fixed t, the map $x_0 \mapsto x(t)$ is a homeomorphism. Therefore, mutatis mutandis, we can ask questions similar to (P_6) about the equation (2).

If f is continuous and bounded (but not necessarily Lipschitz), then by the theorem of Peano, for every $x_0 \in \mathbb{R}^n$, the equation (2) has solutions $x : \mathbb{R} \to \mathbb{R}^n$ with $x(0) = x_0$. But such solutions are no longer unique. Perhaps the necessity of choosing among such solutions can be viewed again as a model of unpredictability or free will, but we think that the equation (1) is more appropriate for modelling the work of the brain than the equation (2) (for reasons visible in [13]), and, of course, given x_0, (1) determines a single forward trajectory. Still, let us add that there exist continuous $f : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x_0 \in \mathbb{R}^n$ there exist solutions $x(t)$ and $\tilde{x}(t)$ of (2) such that $x(0) = \tilde{x}(0) = x_0$ but $x(t) \neq \tilde{x}(t)$ for all $t \neq 0$ (see [6], [16]). *Again we can ask if this non-unicity is true for almost all f.* Let $X(x_0)$ denote the set of all solutions of (2) satisfying $x(0) = x_0$ over the interval $0 \leq t \leq 1$. It is known that the set $X(x_0)$ is connected and of contractible shape (see [2], [12]).

3. Problems about independence

Let G be the additive group of continuous functions $f : 2^\omega \to \mathbb{R}$, where 2^ω denotes the nowhere dense set of Cantor in $[0,1]$. The distance in G is

$$\max_{x \in 2^\omega} |f_1(x) - f_2(x)|.$$

We shall say that f *avoids* a set $N \subseteq \mathbb{R}^n$ iff for any n distinct points $x_1, \ldots, x_n \in 2^\omega$ we have

$$(f(x_1), \ldots, f(x_n)) \notin N.$$

(P_8) Is the set of all functions f which avoid a null set N prevalent in G? [This problem is open already for the case $n = 1$. In this case f avoids N iff $f[2^\omega] \cap N$ is empty. It is also open in the case $n = 2$ and $N = \{(x, x) : x \in \mathbb{R}\}$. In this case f avoids N iff f is injective.] And we ask the same questions for the subgroups of G consisting of all differentiable or Lipschitz functions with appropriate complete metrizations.

Several facts similar to (P_8) are known:

(a) There exist functions in G which avoid a given null set N, and even a countable sequence of null sets $N_i \subseteq \mathbb{R}^{n(i)}$. (This is shown in [10], see also Theorem 3.3 in [7].)

(b) If N is meager in $\mathbb{R}^n$, then the set of functions which avoid N is comeager in G. (This is shown in [11], see also [5].)

(c) Another result related to (P_8) is Proposition 4 in [4].

(d) Such results were applied in [9], [10], [11] and [18] to show the existence of perfect (i.e., nonempty, closed and dense in itself) sets of algebraically independent elements in various groups, fields and general algebras.

Let now $M \subseteq \mathbb{R}^2$ be a set whose complement has two-dimensional Lebesgue measure zero. We shall say that a continuous function $f : 2^\omega \to \mathbb{R}$ (i.e., $f \in G$) is *clever* iff there exists a set $A \subseteq \mathbb{R}$ whose complement has Lebesgue measure zero such that $f[2^\omega] \times A \subseteq M$.

(P_9) Is the set of clever functions prevalent in G? And we ask the same question for subgroups of G with appropriate metrizations as in (P_8).

It is known that:

(a) Clever functions exist in G. (This was shown by H. G. Eggleston [1].)

(b) If M is comeager in $\mathbb{R}^2$ (rather than conull), then the set of all $f \in G$ for which there exist a comeager set $A \subseteq \mathbb{R}$ such that $f[2^\omega] \times A \subseteq M$, is comeager in G. (This was announced in [3].)

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