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Addendum to A. Grothendieck's EGA V

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In EGA V [1], Alexandre Grothendieck reduces the proof of Bertini's Theorem to the following lemma:

1. Lemma (Zariski). *Let k be a field, K an extension of finite type over k, m an integer $\geqq 0$, a_i $(0 \leqq i \leqq m)$ the elements of K such that the transcendence degree of $k(a_0, \ldots, a_m)$ over k is $\geqq 2$. Let $L = K(S_1, \ldots, S_m)$ and $k' = k\left(S_1, \ldots, S_m, a_0 + \sum_1^m a_i S_i\right)$ of L (the S_i being indeterminates). If k is quasi-algebraically closed in K, then k' is quasi-algebraically closed in L.*

Grothendieck omits the proof, referring the reader to the proofs of this lemma's many "brothers" in the literature. The closest statement to the above lemma appears in "Zariski's Collected Papers". It reads:

2. Proposition. *Let K/F be a field of algebraic functions of transcendence degree $r \geqq 2$, let $z_1, \ldots, z_m$ be elements of K such that $F(z)(= F(z_1, \ldots, z_m))$ has transcendence degrees $s \geqq 2$ over F, let $u_1, \ldots, u_m$ be algebraically independent elements over K and let z_u denote the linear form $u_1 z_1 + u_2 z_2 + \cdots + u_m z_m$. If F is quasi-algebraically closed in K, then the field $F(z_u, u)(= F(z_u, u_1, u_2, \ldots, u_m))$ is quasi-algebraically closed in $K(u)$* ([2], page 304).

It does not appear that (1) is a direct consequence of the conclusion of the Proposition, but Zariski's proof of (2) can be modified to prove the Lemma. We do this below for the sake of completeness.

Proof of (1). After a k-linear change in the S_i we may assume that a_0 and a_m are algebraically independent over k. By a specialization argument we obtain $a_0 + \sum_1^{m-1} a_i S_i$ and a_m are algebraically independent over $k(S_1, \ldots, S_{m-1})$. Also, $k(S_1, \ldots, S_{m-1})$ is quasi-algebraically closed in $K(S_1, \ldots, S_{m-1})$. Replacing k by $k(S_1, \ldots, S_{m-1})$ and K by $K(S_1, \ldots, S_{m-1})$, we reduce to the $m = 1$ case.

K is an algebraic extension of $k(a_0, a_1)$. Let K_0 be the separable closure of $k(a_0, a_1)$ in K. Then $k' = k(S, a_0 + a_1 S)$ is quasi-algebraically closed in $K(S)$ if k' is quasi-algebraically closed in $K_0(S)$. Therefore we may now assume that K is separable over $k(a_0, a_1)$.

Let $\bar{K}$ be the least normal extension of $k(a_0, a_1)$ containing K. Then S is algebraically independent over $\bar{K}$. Let $\bar{k}$ denote the algebraic closure of k in $\bar{K}$. Verification of the following claim allows us to reduce further to the case where K is Galois over $k(a_0, a_1)$.

Claim. $\bar{k}(S, a_0 + a_1 S) \cap K(S) = k'$.

Suppose then that $t \in \bar{k}(S, a_0 + a_1 S) \cap K(S)$. Since S and $a_0 + a_1 S$ are algebraically independent over k, they are also algebraically independent over $\bar{k}$. Therefore $t = f/g$, where $f, g \in \bar{k}[S, a_0 + a_1 S]$ and f and g are relatively prime. By insisting that some fixed nonzero coefficient of f is 1, we obtain f and g are unique. Given $\varphi \in \mathrm{Aut}(K'/K)$, $\varphi(t) = t$ since $t \in K(S)$, which implies that the coefficients of f and g are fixed by φ. Thus these coefficients must belong to K and hence they must belong to $K \cap \bar{k} = k$. Hence $t \in k(S, a_0 + a_1 S) = k'$, which proves the claim.

Assume now that K is Galois over $k(a_0, a_1)$. Let T be an indeterminate over $K(S)$. Each automorphism of $K/k(a_0, a_1)$ has a unique extension to $K(s, T)/k(S, T, a_0, a_1)$ and $[K(S, T)/k(S, T, a_0, a_1)] = [K\colon k(a_0, a_1)]$. It follows that $K(S, T)$ is Galois over $k(S, T, a_0, a_1)$. The Galois groups of $K(S, T)/k(S, T, a_0, a_1)$ and $K/k(a_0, a_1)$ are isomorphic and we shall identify each element of the latter with its extension.

Denote by H_S the algebraic closure of $k(S, a_0 + a_1 S)$ in $K(S)$ and by H_T the algebraic closure of $k(T, a_0 + a_1 T)$ in $K(T)$.

Claim. $H_S(T, a_0 + a_1 T) = H_T(S, a_0 + a_1 S)$.

Since both of these fields contain a_0, a_1, S and T, we need only show that the Galois group of $K(S, T)/H_S(T, a_0 + a_1 T)$ coincides with the Galois group of $K(S, T)/H_T(S, a_0 + a_1 S)$. Let G_1 and G_2 denote these two groups. Let φ be the automorphism on $K(S, T)/K$ that interchanges S and T. Then $\varphi^{-1} G_1 \varphi = G_2$. Also, φ commutes with each element of the Galois group of $K(S, T)/k(S, T, a_0, a_1)$, since every element of this group is an extension of an automorphism of $K/k(a_0, a_1)$. Therefore $G_1 = G_2$.

Since k is algebraically closed in K, k is algebraically closed in $K(T)$ and hence in H_T. Since S and $a_0 + a_1 S$ are aglebraically independent over $k(T, a_0 + a_1 T)$, it follows that S and $a_0 + a_1 S$ are algebraically independent over H_T. Thus $k(S, a_0 + a_1 S)$ is algebraically closed in $H_T(S, a_0 + a_1 S) = H_S(T, a_0 + a_1 S)$, which implies $k(S, a_0 + a_1 S) = H_S$. □

References

1. J. Blass, P. Blass, and S. Klasa, *Alexandre Grothendieck's EGA V, Part I: Hyperplane Sections and Conic Projections (1) (Interpretation and rendition of his 'prenotes'),* Ulam Quarterly 1, No. 1 (1992), 62–70.
2. Zariski, O., *Collected Papers,* MIT Press, 1973.