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Cylinderlike Surfaces with Multiple Cylindrical Fibrations

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Introduction

In this paper we wish to continue the investigation of cylinderlike surfaces which we began in [C1] and [C2]. These surfaces are defined as follows.

Definition 1. A cylinderlike surface is an affine, rational, nonsingular surface, $T = \text{Spec } A$, over an algebraically closed field k of characteristic zero having the properties that $A^* = k^*$, Pic T is torsion, and T contains a nonempty subset U which is isomorphic to $\mathbf{A}^1 \times C$ where C is a rational curve. We call a subset such as U a cylindrical open set.

[C2, Th. 4.1] gives an explicit description of the rings whose spectra are cylinderlike surfaces. We wish to determine when two of these surfaces are isomorphic. In order to do this, we are considering a certain type of fibration on a cylinderlike surface.

Definition 2. Suppose $T = \text{Spec } A$ is a cylinderlike surface and $z \in A$. Then $\{V((z-\zeta)) | \zeta \in k\}$ is a cylindrical fibration of T if, for each $\zeta \in k$, the fibre $V((z-\zeta))$ is irreducible and if the complement of a finite set of these fibres is a cylindrical open set.

From a result which is due to Richard Swan and which is given in (1.1) and (1.2) below, we have that every cylinderlike surface has a cylindrical fibration. In [C1] we showed that this fibration is unique for some cylinderlike surfaces but not for others. Clearly, the cylindrical fibration of $\mathbf{A}^2$ is not unique. This paper gives an explicit description of the other cylinderlike surfaces which have multiple cylindrical fibrations.

1. Preliminary Results

In this section we give several general results concerning cylinderlike surfaces which will be used throughout the paper. Suppose $T = \text{Spec } A$ is a cylinderlike surface with cylindrical open subset U and that $T \not\simeq \mathbf{A}^2$. By a result which is due to Richard Swan [S] and whose proof is given in [C2, Prop. 1.1], there are $s, t \in A$; $d_i, R \in \mathbf{Z}^+$; distinct $\alpha_i \in k$; and height one prime ideals P_i of A such that

$$k[s,t] \subset A \subset k\Big[s, t, \frac{1}{\prod_{i=1}^{R}(s-\alpha_i)}\Big], \tag{1.1}$$

$$(s-\alpha_i) = P_i^{d_i}, \tag{1.2}$$

and $U = T - \bigcup_{i=1}^{R} V((s-\alpha_i))$. Thus, $\{V((s-\alpha)) | \alpha \in k\}$ is a cylindrical fibration of T. A further result, the proof of which is given in [C2, Prop. 1.3] and which is also due to Swan [S], gives us that by making U larger if necessary we may assume that, for $i = 1, 2, \dots, R$,

$$d_i > 1. \tag{1.3}$$

By [C2, Prop. 1.4] we have that

$$\text{Pic } T \simeq \coprod_{i=1}^{R} \mathbf{Z}/d_i\mathbf{Z}. \tag{1.4}$$

Furthermore, a result of David Wright's [W], the proof of which can be found in [C2, Prop. 1.2], gives us that, for $i = 1, 2, \dots, R$,

$$V((s-\alpha_i)) \simeq \mathbf{A}^1 \tag{1.5}$$

and, after possibly enlarging U again, there is $\beta_i \in k$ with

$$t - \beta_i \in P_i. \tag{1.6}$$

However, from (1.4) we have that if U has been enlarged so that (1.3) holds, then none of the remaining $V((s-\alpha_i))$ can be eliminated from the complement of U. Thus, (1.6) must also hold. We will let D_i denote the prime divisor corresponding to P_i.

Lemma 1.1 *Suppose (1.1) and (1.2) hold and f is an irreducible element of $k[s,t]$. If $f = s - \alpha_i$ for some $i \in \{1, 2, \dots, R\}$, then the divisor of f is d_iD_i. Otherwise, the divisor of f is $E + \sum_{i=1}^{R} v_{D_i}(f)D_i$ where E is the prime divisor corresponding to a height one prime ideal Q with $Q \cap k[s,t] = (f)k[s,t]$.*

Proof. This follows from (1.2) and the fact that, by (1.1), $k[s,t] \hookrightarrow A$ induces a birational morphism $T \to \text{Spec } k[s,t]$ which is an isomorphism on $U = T - \bigcup_{i=1}^{R} V((s-\alpha_i))$.

Lemma 1.2 *Suppose (1.1) and (1.2) hold and* $z \in A$ *with* $V((z-\zeta))$ *irreducible for all* $\zeta \in k$ *and with* $V((z)) = V((s-\alpha))$ *for some* $\alpha \in k$*. Then there is* $\gamma \in k^*$ *with* $z = \gamma(s-\alpha)$*.*

Proof. By (1.1), $z \in k(s,t)^*$. Thus, by Lemma 1.1, $z = \gamma(s-\alpha)^n$ for some $n \in \mathbf{Z}^+$. Since $V((z-\zeta))$ is irreducible for all $\zeta \in k$, we have $z = \gamma(s-\alpha)$.

Now we fix $l \in \{1, 2, \ldots, R\}$ and, after a linear change of variables, we assume that, in (1.1)-(1.6), $\alpha_l = \beta_l = 0$. By [C2; (2.1)-(2.17), Lemma 2.5, and Lemma 2.7], there is $q \in \mathbf{Z}$ with $q > 1$; for $j = 1, 2, \ldots, q$, there is $u_j \in k[s,t]$; and, for $j = 1, 2, \ldots, q-1$ and $i = 0, 1, \ldots, j-1$, there are $n_j \in \mathbf{Z}^+$, $\lambda_j \in k^*$, and $e_{i,j} \in \mathbf{Z}$ with $e_{0,j} > 0$ and $0 \le e_{i,j} < n_i$ for $i > 0$ such that

$$d_l = n_1 n_2 \cdot \ldots \cdot n_{q-1}, \tag{1.7}$$

$$u_1 = t, \tag{1.8}$$

$$u_{j+1} = u_j^{n_j} - \lambda_j s^{e_{0,j}} \prod_{i=1}^{j-1} u_i^{e_{i,j}}, \tag{1.9}$$

and

$$v_{D_l}(u_{j+1}) > n_j v_{D_l}(u_j). \tag{1.10}$$

Since $e_{0,j} > 0$, we have from (1.8) and (1.9) that there is $\hat{u}_{j+1} \in k[s,t]$ such that

$$u_{j+1} = t^{n_1 n_2 \cdot \ldots \cdot n_j} - s\hat{u}_{j+1}. \tag{1.11}$$

Induction on j gives us that if $0 \le \epsilon_i < n_i$, then $\sum_{i=1}^{j} n_1 n_2 \cdot \ldots \cdot n_{i-1}\epsilon_i < n_1 n_2 \cdot \ldots \cdot n_j$. Thus, by induction on j, together with (1.8), (1.9), and (1.11), we have, for $j = 1, 2, \ldots, q-1$, that

$$\deg_t(\hat{u}_{j+1}) < n_1 n_2 \cdot \ldots \cdot n_j. \tag{1.12}$$

We also have that, for $j = 1, 2, \ldots, q$, there is $v_j \in A$ such that, in A/P_l,

$$\bar{v}_j \in k^* \tag{1.13}$$

for $j = 1, 2, \ldots, q-1$, and

$$A/P_l = k[\bar{v}_q]. \tag{1.14}$$

Lemma 1.3 *Suppose* $T = \mathit{Spec}\ A$ *is a cylinderlike surface with* $T \not\cong \mathbf{A}^2$*. Then there are* $s, t \in A$ *which satisfy (1.1)-(1.6) and for which* $v_{D_i}(t-\beta_i) < d_i$ *for* $i = 1, 2, \ldots, R$*. For this* s *and* t*, we have, in (1.7), that* $n_1 > 1$*.*

Proof. Suppose $l \in \{1, 2, \ldots, R\}$ with $v_{D_i}(t-\beta_i) < d_i$ for $i = 1, 2, \ldots, l-1$ and $v_{D_l}(t-\beta_l) \ge d_l$. We assume that $\alpha_l = \beta_l = 0$ and we let $u_j \in k[s,t]$ and $n_j \in \mathbf{Z}^+$ be as in (1.7)-(1.10). By (1.2) and [C2; (2.1), (2.3), and

(2.4)], n_j is defined as follows. For $j = 0, 1, \ldots, q-1$, there is $\hat{d}_j \in \mathbf{Z}^+$ with $\hat{d}_0 = d_l$, $\hat{d}_j = \gcd(\hat{d}_{j-1}, v_{D_l}(u_j))$, and $n_j = \hat{d}_{j-1}/\hat{d}_j$. By (1.3) and (1.7), there is $k \in \{1, 2, \ldots, q-1\}$ with $n_j = 1$ for $j = 1, 2, \ldots, k-1$ and $n_k > 1$. Thus, for $j = 1, 2, \ldots, k-1$, we have $\hat{d}_j = \hat{d}_{j-1} = d_l$ and we have $\hat{d}_k < d_l$. Therefore, d_l does not divide $v_{D_l}(u_k)$. We write $v_{D_l}(u_k) = nd_l + r$ with $0 < r < d_l$ and let $t' = u_k/s^n$. By (1.2), we have $v_{D_l}(t') = r$ and, since $0 \leq e_{i,j} < n_i = 1$ for $i = 1, 2, \ldots, j-1$, we have, by (1.8) and (1.9), that

$$s^n t' = u_k = t - \sum_{j=1}^{k-1} \lambda_j s^{e_{0,j}}. \tag{1.15}$$

Thus, by (1.1), $k[s, t'] \subset A \subset k\Big[s, t', \frac{1}{\prod_{i=1}^{R}(s-\alpha_i)}\Big]$. Since s is unchanged, conditions (1.2)-(1.5) still hold. Since $v_{D_l}(t') = r > 0$, we have that $t' \in P_l$. By (1.15) we have that for $i \neq l$, there are $\gamma_i \in k^*$ and $\beta_i' \in k$ with $\gamma_i(t' - \beta_i') = t - \beta_i + (s-\alpha_i)G(s, t')$. Thus, by (1.2) and (1.6), $t' - \beta_i' \in P_i$, and, for $i = 1, 2, \ldots, l-1$, $v_{D_i}(t' - \beta_i') = v_{D_i}(t - \beta_i) < d_i$. By induction on $R - l$ we may assume $v_{D_i}(t - \beta_i) < d_i$ for $i = 1, 2, \ldots, R$. In this case, for any $l \in \{1, 2, \ldots, R\}$, if we let $\hat{d}_1 = \gcd(d_l, v_{D_l}(t))$ as above, then $\hat{d}_1 < d_l$ so that $n_1 = d_l/\hat{d}_1 > 1$.

In the following sections we wish to determine R and q for those cylinderlike surfaces which have multiple cylindrical fibrations, and then use this data together with [C2, Th. 4.1] to give an explicit description of the coordinate rings of these surfaces.

2. Determination of R

Throughout the next two sections we will continue to suppose that $T = \text{Spec } A$ is a cylinderlike surface with $T \not\simeq \mathbf{A}^2$. We will also let $s, t, R, d_i, \alpha_i, \beta_i$, and P_i be as in (1.1)-(1.3) and Lemma 1.3, we will assume that $\alpha_1 = \beta_1 = 0$, and we will let q, n_i, u_i, and v_i be as in (1.7)-(1.14) with $l = 1$. From (1.1) and (1.2), we have that $\{V((s-\alpha)) | \alpha \in k\}$ is a cylindrical fibration of T. Suppose that there is a second cylindrical fibration given by $\{V((z-\zeta)) | \zeta \in k\}$ for some $z \in A$ and that $W = T - \bigcup_{i=1}^{n} V((z-\zeta_i))$ is a cylindrical open subset. As in the first section, we have $x, y \in A$; $d_i', R' \in \mathbf{Z}^+$; $\alpha_i', \beta_i' \in k$; and height one prime ideals Q_i of A with the α_i' distinct and with

$$k[x, y] \subset A \subset k\Big[x, y, \frac{1}{\prod_{i=1}^{R'}(x - \alpha_i')}\Big], \tag{2.1}$$

$$(x - \alpha_i') = Q_i^{d_i'}, \tag{2.2}$$

$$\text{Pic } T \simeq \coprod_{i=1}^{R'} \mathbf{Z}/d_i'\mathbf{Z}, \tag{2.3}$$

$$V((x-\alpha_i')) \simeq \mathbf{A}^1, \tag{2.4}$$

and with $y-\beta_i' \in Q_i$, $d_i' > 1$, and $W \subset T - \bigcup_{i=1}^{R'} V((x-\alpha_i'))$. By (2.1), (2.2), and Lemma 1.2, the second fibration is also given by $\{V((x-\alpha'))|\alpha' \in k\}$.

Since the fibration given by x is distinct from the fibration given by s, Lemma 1.2 also gives us that for all $\alpha, \alpha' \in k$, $V((s-\alpha)) \neq V((x-\alpha'))$. Thus, by (1.2) and (2.2),

$$s - \alpha \notin Q_i. \tag{2.5}$$

We let C_i denote the prime divisor corresponding to Q_i.

Proposition 2.1 *If $T \not\simeq \mathbf{A}^2$ is a cylinderlike surface which has multiple cylindrical fibrations and if (1.1)-(1.3) and (2.1)-(2.3) hold, then $R = R' = 1$.*

Proof. By (1.1), for $j = 1, 2, \ldots, R'$, we have $x - \alpha_j' = \frac{\Phi_j(s,t)}{\prod_{i=1}^{R}(s-\alpha_i)^{m_{i,j}}}$ for some $m_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ and some $\Phi_j \in k[s,t]$. By (2.2), the divisor of $x - \alpha_j'$ is $d_j' C_j$. By (2.5), $C_j \neq D_i$ for $i = 1, 2, \ldots, R$. Thus, by Lemma 1.1 there is an irreducible ϕ_j in $k[s,t]$ such that $\Phi_j = \phi_j^{d_j'}$,

$$m_{i,j} d_i = d_j' v_{D_i}(\phi_j), \tag{2.6}$$

$$Q_j \cap k[s,t] = (\phi_j) k[s,t], \tag{2.7}$$

and, the divisor of ϕ_j is $C_j + \sum_{i=1}^{R} v_{D_i}(\phi_j) D_i$. Thus, for $j, l \in \{1, 2, \ldots, R'\}$, we have

$$\frac{\phi_j^{d_j'}}{\prod_{i=1}^{R}(s-\alpha_i)^{m_{i,j}}} = \frac{\phi_l^{d_l'} + (\alpha_l' - \alpha_j') \prod_{i=1}^{R}(s-\alpha_i)^{m_{i,l}}}{\prod_{i=1}^{R}(s-\alpha_i)^{m_{i,l}}}.$$

Since, by (2.5) and (2.7), $(s-\alpha_i)$ divides neither ϕ_j nor ϕ_l, we have $m_{i,j} = m_{i,l}$ and $\phi_j^{d_j'} = \phi_l^{d_l'} + (\alpha_l' - \alpha_j') \prod_{i=1}^{R}(s-\alpha_i)^{m_{i,l}}$.

Suppose $R' > 1$. By (2.5) and (2.7), $\deg_t(\phi_1) > 0$ and $\deg_t(\phi_2) > 0$. Choose $\alpha \in k$ such that $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_R\}$, $\deg(\phi_1(\alpha, t)) > 0$, and $\deg(\phi_2(\alpha, t)) > 0$. Let $F_1(t) = \phi_1(\alpha, t)$ and $F_2(t) = \phi_2(\alpha, t)$. Then there is $\beta \in k^*$ with

$$F_1(t)^{d_1'} = F_2(t)^{d_2'} + \beta \tag{2.8}$$

so that the zeros of F_1 are distinct from the zeros of F_2. Without loss of generality, we assume that $\deg(F_2) \geq \deg(F_1)$. Since $d_2' > 1$, we have $d_2' \deg(F_2) - \deg(F_2) - \deg(F_1) \geq 0$. Thus, $(d_1' - 1)\deg(F_1) + (d_2' - 1)\deg(F_2) > d_1' \deg(F_1) - 1$. But this is impossible since differentiating (2.8) gives us a polynomial of degree $d_1' \deg(F_1) - 1$ with at least $(d_1'-1)\deg(F_1) + (d_2'-1)\deg(F_2)$ zeros. Thus, $R' = 1$ and, by the symmetry of $k[s,t]$ and $k[x,y]$, $R = 1$.

As a consequence of Lemma 2.1, we have, for every cylinderlike surface not isomorphic to $\mathbf{A}^2$, that if (1.1)-(1.3) are satisfied, then the data $R, d_1, d_2, \ldots, d_R$ is unique. If the surface has multiple cylindrical fibrations, then $R = 1$ and d_1 is determined by (1.4). If the cylindrical fibration is unique, then, by 1.1, 1.2, and Lemma 1.1, R is the number of fibres which correspond to a prime ideal which is not principal and, for $i = 1, 2, \ldots, R$, if D_i is the prime divisor corresponding to the fibre $V((s-\alpha_i))$, then d_i is the least positive integer with d_iD_i principal.

3. Determination of q

We continue to suppose that our cylinderlike surface has a second cylindrical fibration and that x,y,R',d_i', α_i', β_i', and Q_i are as in (2.1)-(2.6). Proposition 2.1, together with (1.4) and (2.3), gives us that $d_1 = d_1'$. For the remainder of this paper we will denote d_1 by d, P_1 and Q_1 by P and Q respectively, and D_1 and C_1 by D and C respectively. We will also assume that $\alpha_1' = \beta_1' = 0$. Furthermore, we will let $l(M)$ denote the length of a k-module M.

From the proof of Proposition 2.1 we have $\phi \in k[s,t]$ with ϕ irreducible and

$$x = \frac{\phi^d}{s^{v_D(\phi)}}. \tag{3.1}$$

Let $d' = \gcd(d, v_D(\phi))$, $e = \frac{d}{d'}$, and $e' = \frac{v_D(\phi)}{d'}$. Then $x = \left(\frac{\phi^e}{s^{e'}}\right)^{d'}$ so that, by (2.1),

$$\gcd(d, v_D(\phi)) = 1. \tag{3.2}$$

Lemma 3.1 *There is $m_0 \in \mathbf{Z}^+$, $\eta_0 \in k^*$, and $\phi' \in k[s,t]$ such that $\phi = \eta_0 t^{dm_0} + s\phi'$ with $\deg_t(\phi) = dm_0$ and $v_D(\phi) = m_0 v_D(u_q)$.*

Proof. From [C2, (3.15)] there are $q', c_i \in \mathbf{Z}^{\mathrm{nonneg}}$ with $0 < q' < q$ and $\sum_{i=1}^{q'} c_i v_D(u_i) = c_0 d + 1$. Choose $n \in \mathbf{Z}^+$ with $dn \geq v_D(\phi)$, let $\sigma = dn - v_D(\phi)$, and, for $i = 1, 2, \ldots, q'$, let $\sigma_i = \sigma c_i$. Then $v_D((\prod_{i=1}^{q'} u_i^{\sigma_i})\phi) = dM$ where $M = \sigma c_0 + n$. Let $w = \frac{\left(\prod_{i=1}^{q'} u_i^{\sigma_i}\right)\phi}{s^M}$. By [C2, Lemma 2.6] there are $\zeta_j \in k^*$ and $\mu_{i,j} \in \mathbf{Z}^{\mathrm{nonneg}}$ with $0 \leq \mu_{i,j} < n_i$ for $i = 1, 2, \ldots, q-1$ such that

$$\phi = \sum_{j=1}^{N} \zeta_j s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}}, \tag{3.3}$$

$$\min_j \{v_D(s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}})\} = v_D(\phi), \tag{3.4}$$

and such that if $\mu_{i,j} = \mu_{i,k}$ for $i = 0, 1, \ldots, q$, then $j = k$. For $i > q'$ let $\sigma_i = 0$ and let $J = \{j | v_D(s^{\mu_{0,j}} \prod_{i=1}^{q} u_i^{\mu_{i,j}}) = v_D(\phi)\}$. For $j \notin J$, $\frac{\prod_{i=1}^{q} u_i^{\mu_{i,j}+\sigma_i}}{s^{M-\mu_{0,j}}} \in$

P. Thus, there is $a' \in P$ such that $w = a' + \sum_{j\in J} \zeta_j \frac{\prod_{i=1}^q u_i^{\mu_{i,j}+\sigma_i}}{s^{M-\mu_{0,j}}}$. For $j \in J$, we have, by [C2, Lemma 3.1], that there are $\mu'_{i,j} \in \mathbf{Z}$ with $\mu'_{q,j} = \mu_{q,j}$ such that $\frac{\prod_{i=1}^q u_i^{\mu_{i,j}+\sigma_i}}{s^{M-\mu_{0,j}}} = \prod_{i=1}^q v_i^{\mu'_{i,j}}$. Thus, by (1.13), there are $a \in P$ and $\zeta'_j \in k^*$ with $w = a + F(v_q)$ where $F(v_q) = \sum_{j\in J} \zeta'_j v_q^{\mu_{q,j}}$.

From (2.2) and (2.4) we have that $A/Q \simeq k[X]$. We wish to relate $\deg(F)$ to $\deg(\bar{s})$ in A/Q. Suppose $\hat{m}$ is a maximal ideal of A/Q which contains $\bar{s}$. Let $\pi : A \to A/Q$ and let $m = \pi^{-1}(\hat{m})$. Since $s \in m$, we have, by (1.2), that $P \subset m$. By (1.14), $m = (P, v_q - \zeta)$ for some $\zeta \in k$.

Let $v = \frac{\prod_{i=1}^{q'} u_i^{c_i}}{s^{c_0}}$. Then $v_D(v) = 1$ so $v \in P$. Since $s \notin Q$, we have, by (2.7) together with (1.1) and Proposition 2.1, that $Q = \{\frac{\phi\psi}{s^n} | \psi \in k[s,t]$ and $v_D(\phi\psi) \geq nd\}$. Thus, $w \in Q \subset m$ and $v_q - \zeta$ divides $F(v_q)$. Let $F(v_q) = (v_q-\zeta)^e F'(v_q)$ with $F' \notin m$. Now let $z = \frac{\phi\psi}{s^n} \in Q$. If $v_D(\phi\psi) > nd$, then $z \in P$. If $v_D(\phi\psi) = nd$, then, as above, we can find $L \in \mathbf{Z}^+$ and $\rho_i \in \mathbf{Z}^{\text{nonneg}}$ with $v_D((\prod_{i=1}^{q'} u_i^{\rho_i})\psi) = dL$ so that $v_D(\prod_{i=1}^{q'} u_i^{\sigma_i+\rho_i}) = d(M+L-n)$. Let $w' = \frac{(\prod_{i=1}^{q'} u_i^{\rho_i})\psi}{s^L}$ and $z' = \frac{\prod_{i=1}^{q'} u_i^{\sigma_i+\rho_i}}{s^{M+L-n}}$. By [C2, Lemma 3.1], there are $\sigma'_i \in \mathbf{Z}$ with $z' = \prod_{i=1}^{q-1} v_i^{\sigma'_i}$. By (1.13), $v_i - \theta_i \in m$ for some $\theta_i \in k^*$. Thus, $z' \notin m$. Therefore, $z = \frac{ww'}{z'} = \frac{w'}{z'}(a + F(v_q)) \in ((v_q-\zeta)^e, P)A_m$. Thus, $(v, Q)A_m \subset ((v_q-\zeta)^e, P)A_m$.

On the other hand, by Proposition 2.1, (1.1), and (1.2), we have $P = \{\frac{G(s,t)}{s^n} | v_D(G) > dn\}$. Suppose $\frac{G}{s^n} \in P$. By [C2, Lemma 2.6], there are $\xi_j \in k^*$ and $\nu_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ with $G = \sum_{j=1}^K \xi_j s^{\nu_{0,j}} \prod_{i=1}^q u_i^{\nu_{i,j}}$ and, for $j = 1, 2, \ldots, K$, $v_D(s^{\nu_{0,j}} \prod_{i=1}^q u_i^{\nu_{i,j}}) \geq v_D(G) > dn$. Thus, by [C2, Lemma 3.1], we have $\nu'_{i,j} \in \mathbf{Z}$ with $\nu'_{0,j} > 0$ and $\nu'_{q,j} \geq 0$ such that $\frac{G}{s^n} = \sum_{j=1}^K \xi_j v^{\nu'_{0,j}} \prod_{i=1}^q v_i^{\nu'_{i,j}}$. Since $v_i \notin m$ for $i = 1, 2, \ldots, q-1$, $\frac{G}{s^n} \in (v, Q)A_m$. Since $(v_q-\zeta)^e = \frac{1}{F'(v_q)}(w - a)$, we have $(v, Q)A_m = ((v_q-\zeta)^e, P)A_m$.

Now let $Y = \text{Spec } A/Q$ and let p be the point of Y corresponding to $\hat{m}$. Let $\tilde{m} = m/P$. By [C2, Lemma 3.1], we have $\tau_i \in \mathbf{Z}$ with $s = v^d \prod_{i=1}^{q-1} v_i^{\tau_i}$. Since $v_i \notin m$ for $i = 1, 2, \ldots, q-1$, $v_{p,Y}(\bar{s}) = dv_{p,Y}(\bar{v}) = dl\left(\frac{(A/Q)_{\hat{m}}}{(\bar{v})(A/Q)_{\hat{m}}}\right) = dl\left(\frac{A_m}{(v,Q)A_m}\right) = dl\left(\frac{A_m}{((v_q-\zeta)^e,P)A_m}\right) = dl\left(\frac{(A/P)_{\tilde{m}}}{((v_q-\zeta)^e)(A/P)_{\tilde{m}}}\right) = de$. Thus, $\deg(\bar{s}) = d\deg(F) = d\max_{j\in J}\{\mu_{q,j}\}$. Therefore, from (2.5), we have $0 < \max_{j\in J}\{\mu_{q,j}\}$.

In (3.3), for $j = 1, 2, \ldots, N$, let $\mu_j = \sum_{i=1}^q n_1 n_2 \cdot \ldots \cdot n_{i-1}\mu_{i,j}$. By (1.11) and (1.12), $s^{\mu_{0,j}} \prod_{i=1}^q u_i^{\mu_{i,j}} = s^{\mu_{0,j}} t^{\mu_j} + f_j(s,t)$ with $\deg_t(f_j) < \mu_j$. Suppose $j, l \in \{1, 2, \ldots, N\}$ with $\mu_{0,j} = \mu_{0,l}$ and $\mu_j = \mu_l$. Since $0 \leq \mu_{i,j}, \mu_{i,l} < n_i$ and $\mu_j = \mu_l$, induction on i shows that $\mu_{i,j} = \mu_{i,l}$ for $i = 1, 2, \ldots, q$. Thus, $j = l$ and $\deg_t(\phi) = \max_j\{\mu_j\}$.

We now choose $l \in J$ with $\mu_{q,l} = \max_{j\in J}\{\mu_{q,j}\}$. By (1.7), $d\mu_{q,l} \leq \mu_l \leq \deg_t(\phi)$. Suppose $\deg_t(\phi) > d\mu_{q,l}$. Then there is $\alpha \in k^*$ with $\deg(\phi(\alpha,t)) > d\mu_{q,l}$. Since, by (1.1) and Proposition 2.1, $\rho : T \to \text{Spec } k[s,t]$

is an isomorphism on $T - V((s))$, we have that, in A/Q, $\deg(\bar{s} - \alpha) = l\left(\frac{A/Q}{(\bar{s}-\alpha)A/Q}\right) = l\left(\frac{k[s,t]}{(s-\alpha,\phi)}\right) = \deg(\phi(\alpha,t)) > d\mu_{q,l} = \deg(\bar{s})$. Therefore, $\deg_t(\phi) = d\mu_{q,l}$.

By (2.5) and (2.7), s does not divide ϕ. Let $j \in \{1, 2, \ldots, N\}$ with $\mu_{0,j} = 0$. Then

$$\mu_{1,j} + n_1\mu_{2,j} + n_1n_2\mu_{3,j} + \ldots + d\mu_{q,j} = \mu_j \le \deg_t(\phi) = d\mu_{q,l}. \tag{3.5}$$

In particular, $\mu_{q,j} \le \mu_{q,l}$. Since $l \in J$, we also have by (3.4), that

$$\sum_{i=1}^{q} \mu_{i,j} v_D(u_i) \ge \mu_{0,l} d + \sum_{i=1}^{q} \mu_{i,l} v_D(u_i) \ge \mu_{q,l} v_D(u_q). \tag{3.6}$$

But induction on q together with (1.10) and the fact that $0 \le \mu_{i,j} < n_i$ for $i = 1, 2, \ldots, q-1$, gives us that $\sum_{i=1}^{q-1} \mu_{i,j} v_D(u_i) < v_D(u_q)$. Thus, $\mu_{q,j} = \mu_{q,l}$ and, by (3.5), $\mu_{i,j} = 0$ for $i = 1, 2, \ldots, q-1$. Therefore, we have equality in (3.6) and $\mu_{i,l} = 0$ for $i = 0, 1, \ldots, q-1$. Thus, $j = l$, $\mu_l = d\mu_{q,l}$, and $\phi = \eta_0 t^{dm_0} + s\phi'$ where $\eta_0 = \zeta_l$ and $m_0 = \mu_{q,l}$. Also, by (3.4), $v_D(\phi) = m_0 v_D(u_q)$.

By (2.7), (3.1), and the symmetry of $k[s,t]$ with $k[x,y]$, there is an irreducible f in $k[x,y]$ such that $s = \frac{f^d}{x^{v_C(f)}}$ and

$$P \cap k[x,y] = (f)k[x,y]. \tag{3.7}$$

By (1.1) and Proposition 2.1, we have $f = \frac{F(s,t)}{s^p}$ for some $p \in \mathbf{Z}^{\text{nonneg}}$ and some $F \in k[s,t]$ with F not divisible by s. Thus, by (3.1), $\frac{F^d}{s^{pd}} = f^d = \frac{\phi^{d v_C(f)}}{s^{v_D(\phi)v_C(f)-1}}$. Therefore,

$$v_C(f)v_D(\phi) = pd + 1 \tag{3.8}$$

and we may take

$$f = \frac{\phi^{v_C(f)}}{s^p}. \tag{3.9}$$

Also, by Lemma 1.1 and the symmetry of $k[x,y]$ with $k[s,t]$, there is a prime divisor C' with the divisor of y equal to $C' + v_C(y)C$. Symmetry together with Lemma 3.1 and (1.3) gives us that $\deg_y(f) \ge d1$ so that $C' \ne D$. Thus, by Lemma 1.1, there is an irreducible ψ in $k[s,t]$ with $\psi \ne s$ and $y = \frac{\phi^b\psi}{s^a}$ where $b = v_C(y)$ and $a \in \mathbf{Z}^{\text{nonneg}}$. Since $y \in Q$, we have $b > 0$. Since $C' \ne D$, we have $v_D(y) = 0$ so that $v_D(\psi) = ad - bv_D(\phi)$ and $a > 0$.

Again, by the symmetry of $k[x,y]$ with $k[s,t]$, we have an irreducible g in $k[x,y]$ with $t = \frac{f^{v_D(t)}g}{x^{a'}}$, and with $v_C(g) = a'd - v_D(t)v_C(f)$. Thus, by (3.1), (3.8), and (3.9),

$$g = \frac{\phi^{v_C(g)}t}{s^c} \tag{3.10}$$

with

$$dc = v_C(g)v_D(\phi) + v_D(t). \tag{3.11}$$

We now wish to define a finite sequence $\psi_1, \psi_2, \ldots \psi_r$ of elements of $k[s,t]$ and a finite sequence $h_1, h_2, \ldots, h_r$ of elements of $k[x,y]$. We let

$$\delta_0 = v_D(\phi), \tag{3.12}$$

$$h_1 = y, \tag{3.13}$$

and $\psi_1 = \psi$. From the discussion above (3.10), we have $h_1 = \frac{\phi^{b_1}\psi_1}{s^{a_1}}$ with $a_1, b_1 > 0$ and $v_D(\psi_1) = a_1 d - b_1\delta_0$. Suppose $l \in \mathbf{Z}^+$ and assume that, for $j = 1, 2, \ldots, l$, we have defined $\delta_{j-1} \in \mathbf{Z}^+$, $\psi_j \in k[s,t]$, and $h_j \in k[x,y]$ so that there are $a_j, b_j \in \mathbf{Z}^+$ with

$$h_j = \frac{\phi^{b_j}\psi_j}{s^{a_j}}, \tag{3.14}$$

$$v_D(\psi_j) = a_j d - b_j\delta_0, \tag{3.15}$$

and so that, for $j = 1, 2, \ldots, l-1$, there are $m_j \in \mathbf{Z}^{\text{nonneg}}$, $\eta_j \in k^*$, and $\psi_j' \in k[s,t]$ with

$$\psi_j = \eta_j t^{dm_j} + s\psi_j' \tag{3.16}$$

and

$$v_D(\psi_j) = m_j v_D(u_q). \tag{3.17}$$

We further assume, for $j = 1, 2, \ldots, l-1$, that δ_j, ψ_{j+1}, and h_{j+1} are defined as follows. Let $\delta_j = \gcd(\delta_{j-1}, a_j)$,

$$\nu_j = \frac{\delta_{j-1}}{\delta_j}, \tag{3.18}$$

and

$$\nu_j' = \frac{a_j}{\delta_j}. \tag{3.19}$$

Let $\epsilon_{j,j}' = \nu_j'$. Since $\gcd(\nu_i, \nu_i') = 1$, for $i = 1, 2, \ldots, j-1$, there are integers $\epsilon_{i,j}$ and $\epsilon_{i,j}'$ such that $0 \le \epsilon_{i,j} < \nu_i$ and $\epsilon_{i+1,j}' = \epsilon_{i,j}'\nu_i + \epsilon_{i,j}\nu_i'$. Let $\epsilon_{0,j} = \max\{\epsilon_{1,j}', 0\}$ and $\epsilon_{0,j}' = \epsilon_{0,j} - \epsilon_{1,j}'$. Then

$$\epsilon_{0,j}'\delta_0 + \nu_j a_j = \epsilon_{0,j}\delta_0 + \sum_{i=1}^{j-1}\epsilon_{i,j}a_i. \tag{3.20}$$

Let $\mu_j = \max\{\epsilon_{0,j}d - \epsilon_{0,j}'d + \sum_{i=1}^{j-1}\epsilon_{i,j}b_i - \nu_j b_j, 0\}$ and $\mu_j' = \mu_j + \epsilon_{0,j}'d - \epsilon_{0,j}d + \nu_j b_j - \sum_{i=1}^{j-1}\epsilon_{i,j}b_i$. Then, by (3.20), we have $(\mu_j' - \mu_j)\delta_0 = d(\sum_{i=1}^{j-1}\epsilon_{i,j}a_i - \nu_j a_j) + \nu_j b_j\delta_0 - \sum_{i=1}^{j-1}\epsilon_{i,j}b_i\delta_0$ so that, by (3.12) and (3.15), we have

$v_D(\phi^{\mu'_j}\psi_j^{\nu_j}) = v_D(\phi^{\mu_j}\prod_{i=1}^{j-1}\psi_i^{\epsilon_{i,j}})$. Note that we only need (3.15) to hold for $j = 1, 2, \ldots, l-1$. By Lemma 3.1 and (3.17), dividing both sides of this equation by $v_D(u_q)$ gives $\mu'_j m_0 + \nu_j m_j = \mu_j m_0 + \sum_{i=1}^{j-1}\epsilon_{i,j}m_i$. Thus, by Lemma 3.1 and (3.16), there is $\gamma_j \in k^*$ such that s divides $\phi^{\mu'_j}\psi_j^{\nu_j} - \gamma_j\phi^{\mu_j}\prod_{i=1}^{j-1}\psi_i^{\epsilon_{i,j}}$. We assume that ψ_{j+1} is defined by

$$\phi^{\mu'_j}\psi_j^{\nu_j} - \gamma_j\phi^{\mu_j}\prod_{i=1}^{j-1}\psi_i^{\epsilon_{i,j}} = s^{e_j}\psi_{j+1} \tag{3.21}$$

where $e_j > 0$ and s does not divide ψ_{j+1} and that h_{j+1} is defined by

$$h_{j+1} = x^{\epsilon'_{0,j}}h_j^{\nu_j} - \gamma_j x^{\epsilon_{0,j}}\prod_{i=1}^{j-1}h_i^{\epsilon_{i,j}}. \tag{3.22}$$

We must now either end the sequences with ψ_l and h_l or continue the sequences by first showing that ψ_l satisfies properties (3.16) and (3.17) so that we may define δ_l, ψ_{l+1}, and h_{l+1} as in (3.18)-(3.22) and then showing that ψ_{l+1}, and h_{l+1} satisfy properties (3.14) and (3.15). To determine which course to follow we will use the following lemmas.

Lemma 3.2 *For $i = 1, 2, \ldots, l$, let $\tau_i \in \mathbf{Z}^{\text{nonneg}}$. Then there are $\rho, \rho_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ and $\xi_j \in k^*$ with $0 \le \rho_{i,j} < \nu_i$ for $i = 1, 2, \ldots, l-1$ such that $x^\rho\prod_{i=1}^{l}h_i^{\tau_i} = \sum_{j=1}^{n}\xi_j x^{\rho_{0,j}}\prod_{i=1}^{l}h_i^{\rho_{i,j}}$.*

Proof. Suppose $k \in \{1, 2, \ldots, l-1\}$ with $\tau_k \ge \nu_k$ and $\tau_i < \nu_i$ for $i = k+1, \ldots, l-1$. Let $r \in \{k+1, \ldots, l\}$ such that $\tau_i + 1 = \nu_i$ for $i = k+1, \ldots, r-1$ and either $\tau_r + 1 < \nu_r$ or $r = l$. Let $\epsilon = \sum_{j=k}^{r-1}\epsilon'_{0,j}$ and, for $j = k, \ldots, r-1$, let $\epsilon_j = \epsilon_{0,j} + \sum_{i=j+1}^{r-1}\epsilon'_{0,i}$. Then induction on $r-k$ together with (3.22) shows that

$$x^\epsilon h_k^{\nu_k}h_{k+1}^{\tau_{k+1}}\cdot\ldots\cdot h_{r-1}^{\tau_{r-1}} = h_r + \sum_{j=k}^{r-1}\gamma_j x^{\epsilon_j}\Big(\prod_{i=1}^{j-1}h_i^{\epsilon_{i,j}}\Big)\Big(\prod_{i=j+1}^{r-1}h_i^{\tau_i}\Big).$$

Thus, induction on τ_k and k gives the desired result.

To decide whether or not to end the sequences with ψ_l and h_l we consider g from (3.10). By Lemma 3.2, we have $\xi_j \in k^*$ and $\rho, \rho_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ with

$$x^\rho g = \sum_{j=1}^{n}\xi_j x^{\rho_{0,j}}\prod_{i=1}^{l}h_i^{\rho_{i,j}}, \tag{3.23}$$

$0 \le \rho_{i,j} < \nu_i$ for $i = 1, 2, \ldots, l-1$, and such that if $\rho_{i,j} = \rho_{i,k}$ for $i = 0, 1, \ldots, l$, then $j = k$. Thus, by (3.1), (3.10), (3.12), and (3.14), we have

$$\frac{\phi^{\rho d + v_C(g)}t}{s^{\rho\delta_0 + c}} = \sum_{j=1}^{n}\xi_j\frac{\phi^{R_j}\prod_{i=1}^{l}\psi_i^{\rho_{i,j}}}{s^{E_j}} \tag{3.24}$$

where

$$R_j = \rho_{0,j}d + \sum_{i=1}^{l} \rho_{i,j}b_i \tag{3.25}$$

and

$$E_j = \rho_{0,j}\delta_0 + \sum_{i=1}^{l} \rho_{i,j}a_i. \tag{3.26}$$

We let

$$M_l = \max_j\{E_j\} - \rho\delta_0 - c. \tag{3.27}$$

Lemma 3.3 *M_l is independent of the choice of ρ and the $\rho_{i,j}$.*

Proof. Suppose $x^\rho g = \sum_{j=1}^n \xi_j x^{\rho_{0,j}} \prod_{i=1}^l h_i^{\rho_{i,j}}$ and $x^\tau g = \sum_{j=1}^m \zeta_j x^{\tau_{0,j}} \prod_{i=1}^l h_i^{\tau_{i,j}}$ as in (3.23) and that $\rho \geq \tau$. Then $x^\rho g = \sum_{j=1}^m \zeta_j x^{\tau_{0,j}+\rho-\tau} \prod_{i=1}^l h_i^{\tau_{i,j}}$. As in (1.11) and (1.12), induction on j together with (3.13), (3.22), and the fact that $0 \leq \epsilon_{i,j} < \nu_i$, gives us that, for $j = 1, 2, \ldots, l-1$, if $\epsilon'_j = \sum_{i=1}^j \epsilon'_{0,i}\nu_{i+1} \cdot \ldots \cdot \nu_j$, then there is $h'_{j+1} \in k[x,y]$ with

$$h_{j+1} = x^{\epsilon'_j} y^{\nu_1\nu_2\cdot\ldots\cdot\nu_j} + h'_{j+1}(x,y) \tag{3.28}$$

and

$$\deg_y(h'_{j+1}) < \nu_1\nu_2 \cdot \ldots \cdot \nu_j. \tag{3.29}$$

For $j = 1, 2, \ldots, n$, let $\rho_j = \sum_{i=1}^l \nu_1\nu_2 \cdot \ldots \cdot \nu_{i-1}\rho_{i,j}$ and $\rho'_j = \rho_{0,j} + \sum_{i=1}^l \rho_{i,j}\epsilon'_i$. Then $x^{\rho_{0,j}} \prod_{i=1}^l h_i^{\rho_{i,j}} = x^{\rho'_j} y^{\rho_j} + H_j(x,y)$ with $\deg_y(H_j) < \rho_j$. Suppose $j, k \in \{1, 2, \ldots, n\}$ with $\rho_j = \rho_k$ and $\rho'_j = \rho'_k$. Since $\rho_j = \rho_k$, induction on i and the fact that $0 \leq \rho_{i,j} < \nu_i$ give us that $\rho_{i,j} = \rho_{i,k}$ for $i = 1, 2, \ldots, l$. Thus, $\rho'_j = \rho'_k$ implies $\rho_{0,j} = \rho_{0,k}$. Therefore, $j = k$ and $\deg_y(g) = \max_j\{\rho_j\}$. Similarly, if we let $\tau_j = \sum_{i=1}^l \nu_1\nu_2 \cdot \ldots \cdot \nu_{i-1}\tau_{i,j}$ and $\tau'_j = \tau_{0,j} + \rho - \tau + \sum_{i=1}^l \tau_{i,j}\epsilon'_i$, then $\tau_j = \tau_k$ and $\tau'_j = \tau'_k$ imply $j = k$.

Let $g = G(x)y^e + G'(x,y)$ with $\deg_y(G') < e$. Then there is a unique $k \in \{1, 2, \ldots, n\}$ and a unique $r \in \{1, 2, \ldots, m\}$ with $\rho_k = \tau_r = e$ and $\rho'_k = \tau'_r = \deg(G) + \rho$. As above, $\rho_{i,k} = \tau_{i,r}$ for $i = 1, 2, \ldots, l$ and $\rho_{0,k} = \tau_{0,r} + \rho - \tau$ so that $\sum_{j \neq k} \xi_j x^{\rho_{0,j}} \prod_{i=1}^l h_i^{\rho_{i,j}} = \sum_{j \neq r} \zeta_j x^{\tau_{0,j}+\rho-\tau} \prod_{i=1}^l h_i^{\tau_{0,j}}$. Thus, induction on $\deg(G)$ and e gives us that $\max_j\{\rho_{0,j}\delta_0 + \sum_{i=1}^l \rho_{i,j}a_i\} - \rho\delta_0 - c = \max_j\{\tau_{0,j}\delta_0 + \sum_{i=1}^l \tau_{i,j}a_i\} - \tau\delta_0 - c$.

By (3.24) and (3.27), $M_l \geq 0$. If $M_l = 0$, then we end the sequences with ψ_l and h_l. If $M_l > 0$, then the following two lemmas will allow us to continue the sequences.

Lemma 3.4 *If $M_l > 0$, then there are $m_l \in \mathbf{Z}^{\text{nonneg}}$, $\eta_l \in k^*$, and $\psi_l \in k[s,t]$ with $\psi_l = \eta_l t^{dm_l} + s\psi'_l$ and $v_D(\psi_l) = m_l v_D(u_q)$.*

Proof. Using the notation of (3.23)-(3.27), we suppose $M_l > 0$. Let

$$J_l = \{k | E_k = \max_j \{E_j\}\}. \tag{3.30}$$

Then, by (3.24) and (3.27), s divides $\sum_{j \in J} \xi_j \phi^{R_j} \prod_{i=1}^{l} \psi_i^{\rho_{i,j}}$. Let $\psi_l = s\psi_l'(s,t) + \psi_{l,0}(t)$. By Lemma 3.1 and (3.16),

$$0 = \sum_{j \in J} \xi_j' t^{R_j'} (\psi_{l,0}(t))^{\rho_{l,j}} \tag{3.31}$$

where $\xi_j' \in k^*$ and $R_j' = R_j dm_0 + \sum_{i=1}^{l-1} \rho_{i,j} dm_i$. Suppose $j, k \in J_l$ with $\rho_{l,j} = \rho_{l,k}$. Then, by (3.12), (3.18), (3.19), and (3.26), $\rho_{0,j}\nu_1\nu_2 \cdot \ldots \cdot \nu_{l-1}\delta_{l-1} + \sum_{i=1}^{l-2} \rho_{i,j}\nu_i'\nu_{i+1} \cdot \ldots \cdot \nu_{l-1}\delta_{l-1} + \rho_{l-1,j}\nu_{l-1}'\delta_{l-1} = \rho_{0,k}\nu_1\nu_2 \cdot \ldots \cdot \nu_{l-1}\delta_{l-1} + \sum_{i=1}^{l-2} \rho_{i,k}\nu_i'\nu_{i+1} \cdot \ldots \cdot \nu_{l-1}\delta_{l-1} + \rho_{l-1,k}\nu_{l-1}'\delta_{l-1}$. Thus, since $\gcd(\nu_{l-1}, \nu_{l-1}') = 1$, we have that ν_{l-1} divides $\rho_{l-1,k} - \rho_{l-1,j}$. Since $0 \leq \rho_{l-1,j}, \rho_{l-1,k} < \nu_i$, we have $\rho_{l-1,k} = \rho_{l-1,j}$. Induction on $l - i$ shows that $\rho_{i,k} = \rho_{i,j}$ for $i = 0, 1, \ldots, l$ so that $j = k$. Thus, there is a unique element k of J_l with $\rho_{l,k} = \min_{j \in J}\{\rho_{l,j}\}$. Therefore, by (3.31), $\psi_{l,0}(t)$ divides $t^{R_k'}$. From (3.21) and the discussion above (3.10), s does not divide ψ_l. Thus, $\psi_{l,0}(t) = \eta_l t^{m_l'}$ for some $\eta_l \in k^*$ and $m_l' \in \mathbf{Z}^{\text{nonneg}}$. Therefore, by (3.31) there is $j \in J_l$ with $\rho_{l,j} > \rho_{l,k}$ and

$$R_j dm_0 + \sum_{i=1}^{l-1} \rho_{i,j} dm_i + \rho_{l,j} m_l' = R_k dm_0 + \sum_{i=1}^{l-1} \rho_{i,k} dm_i + \rho_{l,k} m_l'. \tag{3.32}$$

By (3.12), (3.15), (3.25), and (3.26), we have $v_D(\phi^{R_j} \prod_{i=1}^{l} \psi_i^{\rho_{i,j}}) = v_D(s^{E_j})$ $= v_D(\phi^{R_k} \prod_{i=1}^{l} \psi_i^{\rho_{i,k}})$. Thus, by Lemma 3.1 and (3.17), multiplying (3.32) by $v_D(u_q)$ qives us $(\rho_{l,j} - \rho_{l,k}) m_l' v_D(u_q) = (\rho_{l,j} - \rho_{l,k}) d v_D(\psi_l)$. By (3.2) and Lemma 3.1, $\gcd(d, v_D(u_q)) = 1$. Thus, there is $m_l \in \mathbf{Z}^{\text{nonneg}}$ with $m_l' = dm_l$ and $v_D(\psi_l) = m_l v_D(u_q)$.

Lemma 3.5 *If $M_l > 0$, and if ψ_{l+1} and h_{l+1} are defined as in (3.18)-(3.22), then there are $a_{l+1}, b_{l+1} \in \mathbf{Z}^+$ with $h_{l+1} = \frac{\phi^{b_{l+1}} \psi_{l+1}}{s^{a_{l+1}}}$ and $v_D(\psi_{l+1})$ $= a_{l+1} d - b_{l+1} \delta_0$.*

Proof. By (3.22), (3.1), (3.14), (3.20), and (3.21), we have $h_{l+1} = \frac{\phi^{b_{l+1}} \psi_{l+1}}{s^{a_{l+1}}}$ where $b_{l+1} = \min\{\epsilon_{0,l}' d + \nu_l b_l, \epsilon_{0,l} d + \sum_{i=1}^{l-1} \epsilon_{i,l} b_i\}$ and

$$a_{l+1} = \epsilon_{0,l}' \delta_0 + \nu_l a_l - e_l. \tag{3.33}$$

Since $a_l > 0$, we have, by (3.20), that $\epsilon_{i,l} > 0$ for some $i \in \{0, 1, \ldots, l-1\}$. Thus, since $b_i > 0$ for $i = 1, 2, \ldots, l$, we have $b_{l+1} > 0$.

By (3.22), $h_{l+1} \in A$ and, thus, $v_D(\psi_{l+1}) \geq a_{l+1} d - b_{l+1} \delta_0$. Suppose $v_D(\psi_{l+1}) > a_{l+1} d - b_{l+1} \delta_0$. Then $v_D(h_{l+1}) > 0$ and, by (3.7), f divides

h_{l+1}. The symmetry of $k[x,y]$ with $k[s,t]$ gives us that, as in Lemma 3.1, d divides $\deg_y(f)$. By (3.2), (3.12), and (3.18), we have that $\gcd(d,\nu_i)=1$. Thus, by (1.3), (3.28), and (3.29), $\deg_y(f) < \deg_y(h_{l+1}) = \nu_1\nu_2\cdot\ldots\cdot\nu_l$. By Lemma 3.2, we have $\zeta_j \in k^*$ and $\sigma,\sigma_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ with $x^\sigma f = \sum_{j=1}^m \zeta_j x^{\sigma_{0,j}} \prod_{i=1}^{l+1} h_i^{\sigma_{i,j}}$, with $0 \le \sigma_{i,j} < \nu_i$ for $i=1,2,\ldots,l$, and such that if $\sigma_{i,j}=\sigma_{i,k}$ for $i=0,1,\ldots,l+1$, then $j=k$. For $j=1,2,\ldots,m$, let $\sigma_j = \sum_{i=1}^{l+1}\nu_1\cdot\ldots\cdot\nu_{i-1}\sigma_{i,j}$. As in the proof of Lemma 3.3, $\deg_y(f)=\max_j\{\sigma_j\}$. Thus, $\sigma_{l+1,j}=0$ for $j=1,2,\ldots,m$. By (3.1), (3.9), and (3.14), we have

$$\frac{\phi^{\sigma d+v_C(f)}}{s^{\sigma\delta_0+p}} = \sum_{j=1}^m \zeta_j \frac{\phi^{Q_j}\prod_{i=1}^l \psi_i^{\sigma_{i,j}}}{s^{F_j}} \tag{3.34}$$

where $Q_j = \sigma_{0,j}d + \sum_{i=1}^l \sigma_{i,j}b_i$ and $F_j = \sigma_{0,j}\delta_0 + \sum_{i=1}^l \sigma_{i,j}a_i$. We let $L=\max_j\{F_j\}$ and $I=\{j|F_j=L\}$. Suppose $j,k\in I$. Since $\sigma_{l+1,j} = \sigma_{l+1,k}=0$, we have, as in the proof of Lemma 3.4, that $j=k$. Thus, there is a single element in I and s does not divide $\sum_{j\in I}\zeta_j\phi^{Q_j}\prod_{i=1}^l\psi_i^{\sigma_{i,j}}$. Therefore, $\phi^{\sigma d+v_C(f)} = \sum_{j=1}^m \zeta_j s^{L-F_j}\phi^{Q_j}\prod_{i=1}^l \psi_i^{\sigma_{i,j}}$ and $L=\sigma\delta_0+p$. Thus, by Lemma 3.1, (3.16), and Lemma 3.4, we have that if $I=\{k\}$, then $(\sigma d + v_C(f))dm_0 = Q_k dm_0 + \sum_{i=1}^l \sigma_{i,k}dm_i$. But this is impossible since, by (3.8), (3.12), and Lemma 3.1, $(\sigma d+v_C(f))m_0 v_D(u_q) = d(\sigma\delta_0+p)+1$ and, by the definitions of Q_k, F_k, and I together with Lemma 3.1, (3.15), (3.17), and Lemma 3.4, $(Q_k m_0 + \sum_{i=1}^l \sigma_{i,k}m_i)v_D(u_q) = dL$. Thus, $v_D(\psi_{l+1}) = a_{l+1}d - b_{l+1}\delta_0$ and $a_{l+1}>0$.

Lemma 3.6 *The sequences defined above must eventually end.*

Proof. Suppose we have defined $h_1,\ldots,h_{l+1}$ as in (3.13) and (3.22). Let $\rho_i \in \mathbf{Z}^{\text{nonneg}}$ with $0\le\rho_i<\nu_i$ for $i=1,2,\ldots,l$ and $\rho_{l+1}>0$. Then, by (3.22),

$$x^{\rho_0}\prod_{i=1}^{l+1} h_i^{\rho_i} = \sum_{k=0}^{\rho_{l+1}} \zeta_k x^{r_k}\Big(\prod_{i=1}^{l-1} h_i^{\rho_i + k\epsilon_{i,l}}\Big) h_l^{\rho_l+\nu_l(\rho_{l+1}-k)}$$

where $\zeta_k\in k^*$ and $r_k = \rho_0 + \epsilon'_{0,l}(\rho_{l+1}-k) + k\epsilon_{0,l}$. Thus, retaining the term corresponding to $k=0$ and applying Lemma 3.2 to the terms corresponding to $k=1,2,\ldots,\rho_{l+1}$, we have $\tau_{i,j}\in\mathbf{Z}^{\text{nonneg}}$ such that $0\le\tau_{i,j}<\nu_i$ for $i=1,2,\ldots,l-1$ and $x^{\rho_0}\prod_{i=1}^{l+1}h_i^{\rho_i} = \sum_{j=1}^n \xi_j x^{\tau_{0,j}}\prod_{i=1}^l h_i^{\tau_{i,j}}$ with $\tau_{0,1} = \rho_0+\epsilon'_{0,l}\rho_{l+1}$, with $\tau_{i,1}=\rho_i$ for $i=1,2,\ldots,l-1$, and with $\tau_{l,1}=\rho_l+\nu_l\rho_{l+1}$. Thus, by (3.33), $\max_j\{\tau_{0,j}\delta_0+\sum_{i=1}^l\tau_{i,j}a_i\} \ge (\rho_0+\epsilon'_{0,l}\rho_{l+1})\delta_0 + \sum_{i=1}^{l-1}\rho_i a_i + (\rho_l+\nu_l\rho_{l+1})a_l = \rho_0\delta_0 + \sum_{i=1}^{l+1}\rho_i a_i + \rho_{l+1}e_l > \rho_0\delta_0 + \sum_{i=1}^{l+1}\rho_i a_i$.

We now write $x^\rho g = \sum_{j=1}^n \xi_j x^{\rho_{0,j}}\prod_{i=1}^{l+1}h_i^{\rho_{i,j}}$ as in (3.23) and define M_{l+1} and J_{l+1} as in (3.27) and (3.30). If $M_{l+1}=0$, the sequences end with ψ_{l+1} and h_{l+1}. If $M_{l+1}>0$, then s divides $\sum_{j\in J_{l+1}}\xi_j\phi^{R_j}\prod_{i=1}^{l+1}\psi_i^{\rho_{i,j}}$ so that J_{l+1} contains at least two elements. As in the proof of Lemma 3.4, if $j,k\in J_{l+1}$ with $j\ne k$, then $\rho_{l+1,j}\ne\rho_{l+1,k}$. Thus, there is $k\in$

J_{l+1} with $\rho_{l+1,k} > 0$. Then from the argument above we have $x^\rho g = \sum_{j=1}^{n'} \xi_j' x^{\tau_{0,j}} \prod_{i=1}^{l} h_i^{\tau_{i,j}}$ with $M_l = \max_j\{\tau_{0,j}\delta_0 + \sum_{i=1}^{l} \tau_{i,j} a_i\} - \rho\delta_0 - c > \rho_{0,k}\delta_0 + \sum_{i=1}^{l+1} \rho_{i,k} a_i - \rho\delta_0 - c = M_{l+1} > 0$. Therefore, there must be $r \in \mathbf{Z}^+$ with $M_r = 0$ and the sequences end with ψ_r and h_r.

Now that we have defined our sequences we are ready to use them to determine q for cylinderlike surfaces with multiple cylindrical fibrations.

Propostion 3.1 *If $T = Spec\ A$ is a cylinderlike surface with multiple cylindrical fibrations, if $T \not\cong \mathbf{A}^2$, and if $s, t \in A$ and $d_i \in \mathbf{Z}^+$ satisfy (1.1)-(1.3) and Lemma 1.3, then in (1.7), we have $q = 2$ and, in (1.10), we have $v_D(u_2) = dv_D(t) + 1$.*

Proof. Suppose the above sequences end with ψ_r and h_r and that ψ_j and h_j satisfy (3.14) and (3.15) for $j = 1, 2, \ldots, r$ and (3.16) and (3.17) for $j = 1, 2, \ldots, r-1$. As in (3.23), we write $x^\rho g = \sum_{j=1}^{n} \xi_j x^{\rho_{0,j}} \prod_{i=1}^{r} h_i^{\rho_{i,j}}$ and define M_r and J_r as in (3.27) and (3.30). Since the sequences end with ψ_r and h_r, we have $M_r = 0$. Let $M = \max_j\{E_j\}$ where E_j is given by (3.26). Then $M = \rho\delta_0 + c$ and, in (3.24), $\phi^{\rho d + v_C(g)} t = \sum_{j=1}^{n} \xi_j s^{M - E_j} \phi^{R_j} \prod_{i=1}^{r} \psi_i^{\rho_{i,j}}$. Let $\psi_r = s\psi_r'(s,t) + \psi_{r,0}(t)$. By Lemma 3.1 and (3.16), we have

$$\eta t^R = \sum_{j \in J_r} \xi_j' t^{R_j'} \Big(\psi_{r,0}(t)\Big)^{\rho_{r,j}} \tag{3.35}$$

where $\eta, \xi_j' \in k^*$, $R = (\rho d + v_C(g)) dm_0 + 1$, and $R_j' = R_j dm_0 + \sum_{i=1}^{r-1} \rho_{i,j} dm_i$. For $j \in J_r$ we have, by (3.11), (3.12), (3.15), (3.25), and (3.26), that

$$v_D(\phi^{\rho d + v_C(g)} t) = v_D(s^{\rho\delta_0 + c}) = v_D(s^M) = v_D(\phi^{R_j} \prod_{i=1}^{r} \psi_i^{\rho_{i,j}}). \tag{3.36}$$

By (1.8) and (1.10), we have that $v_D(u_q)$ does not divide $v_D(t)$. Thus, by Lemma 3.1, (3.17), and (3.36), we have $\rho_{r,j} > 0$. Therefore, by (3.35), we have $\psi_{r,0} = \eta_r t^{m_r'}$ for some $\eta_r \in k^*$ and $m_r' \in \mathbf{Z}^{\text{nonneg}}$ and we have $k \in J_r$ with

$$\rho_{r,k} m_r' = R - R_k' = (\rho d + v_C(g) - R_k) dm_0 - \sum_{i=1}^{r-1} \rho_{i,k} dm_i + 1. \tag{3.37}$$

Multiplying (3.37) by $v_D(u_q)$ gives, by Lemma 3.1, (3.17), and (3.36), that

$$\rho_{r,k} m_r' v_D(u_q) = d(\rho_{r,k} v_D(\psi_r) - v_D(t)) + v_D(u_q). \tag{3.38}$$

By [C2, Lemma 2.6], we may write $\psi_r = \sum_{j=1}^{N} \theta_j s^{\tau_{0,j}} \prod_{i=1}^{q} u_i^{\tau_{i,j}}$ with $\theta_j \in k^*$ and

$$v_D(\psi_r) \leq \tau_{0,j} d + \sum_{i=1}^{q} \tau_{i,j} v_D(u_i) \tag{3.39}$$

for $j = 1, 2, \ldots, N$. By (1.11) and (1.12), there is $l \in \{1, 2, \ldots, N\}$ with $\tau_{0,l} = 0$ and $m'_r = \sum_{i=1}^{q} n_1 n_2 \cdot \ldots \cdot n_{i-1}\tau_{i,l}$. By (3.37), $\gcd(d, m'_r) = 1$ and, by Lemma 1.3, $n_1 > 1$. Thus, by (1.7), $\tau_{1,l} > 0$. Also, from (3.38) and (3.39), we have $\rho_{r,k} v_D(u_q)(\tau_{1,l} + n_1\tau_{2,l} + \ldots + d\tau_{q,l}) \leq d\rho_{r,k}\sum_{i=1}^{q}\tau_{i,l}v_D(u_i) - dv_D(t) + v_D(u_q)$. Since $u_1 = t$, we have

$$(\rho_{r,k}\tau_{1,l} - 1)(v_D(u_q) - dv_D(t)) \leq \rho_{r,k}\sum_{i=2}^{q-1}\tau_{i,l}(dv_D(u_i) - n_1 \cdot \ldots \cdot n_{i-1}v_D(u_q)). \tag{3.40}$$

By (1.7) and (1.10), we have

$$dv_D(u_1) < n_2 \cdot \ldots \cdot n_{q-1}v_D(u_2) < \ldots < n_{q-1}v_D(u_{q-1}) < v_D(u_q) \tag{3.41}$$

and so, for $i = 1, 2, \ldots, q-1$, we have $dv_D(u_i) < n_1 \cdot \ldots \cdot n_{i-1}v_D(u_q)$. Thus, (3.40) is possible only if $\rho_{r,k} = \tau_{1,l} = 1$, $\tau_{i,l} = 0$ for $i = 2, \ldots, q-1$, and we have equality in (3.39). Thus, letting $m_r = \tau_{q,l}$ gives us

$$\psi_r = \eta_r t^{dm_r+1} + s\psi'_r \tag{3.42}$$

and

$$v_D(\psi_r) = m_r v_D(u_q) + v_D(t). \tag{3.43}$$

As in the proof of Lemma 3.5, we have $\zeta_j \in k^*$ and $\sigma, \sigma_{i,j} \in \mathbf{Z}^{\text{nonneg}}$ such that if $\sigma_{i,j} = \sigma_{i,k}$ for $i = 0, 1, \ldots, r$, then $j = k$ and such that (3.34) holds for $l = r$ where Q_j, F_j, L, and I are defined as in the discussion following (3.34). By Lemma 3.1, (3.16) and (3.42), we have $\phi^{Q_j}\prod_{i=1}^{r}\psi_i^{\sigma_{i,j}} = \zeta'_j t^{Q'_j} + s\Psi_j(s,t)$ where $\zeta'_j \in k^*$, $\Psi_j \in k[s,t]$, and $Q'_j = Q_j dm_0 + \sum_{i=1}^{r-1}\sigma_{i,j}dm_i + \sigma_{r,j}(dm_r + 1)$. Suppose $k, l \in I$ with $Q'_k = Q'_l$. As before, we have $v_D(\phi^{Q_k}\prod_{i=1}^{r}\psi_i^{\sigma_{i,k}}) = v_D(s^L) = v_D(\phi^{Q_l}\prod_{i=1}^{r}\psi_i^{\sigma_{i,l}})$. Thus, by Lemma 3.1, (3.17), and (3.43), multiplying the equation $Q'_k = Q'_l$ by $v_D(u_q)$ gives us $(\sigma_{r,k} - \sigma_{r,l})(v_D(u_q) - dv_D(t)) = 0$. Since, from (3.41), $v_D(u_q) > dv_D(t)$, we have $\sigma_{r,k} = \sigma_{r,l}$ and, as in the proof of Lemma 3.4, $k = l$. Thus, s does not divide $\sum_{j\in I}\phi^{Q_j}\prod_{i=1}^{r}\psi_i^{\sigma_{i,j}}$ and, therefore, $L = \sigma\delta_0 + p$ and $\phi^{\sigma d + v_C(f)} = \sum_{j=1}^{m}\zeta_j s^{L-F_j}\phi^{Q_j}\prod_{i=1}^{r}\psi_i^{\sigma_{i,j}}$. By Lemma 3.1, (3.16), and (3.42), there is $k \in I$ with $(\sigma d + v_C(f))dm_0 = Q_k dm_0 + \sum_{i=1}^{r-1}\sigma_{i,k}dm_i + \sigma_{r,k}(dm_r + 1)$. Therefore, d divides $\sigma_{r,k}$. Multiplying the above equation by $v_D(u_q)$, we now have, by Lemma 3.1, (3.17), and (3.43), that

$$d(\sigma d + v_C(f))v_D(\phi) = d(v_D(\phi^{Q_k}\prod_{i=1}^{r}\psi_i^{\sigma_{i,k}})) + \sigma_{r,k}(v_D(u_q) - dv_D(t)).$$

By (3.8) and (3.12), this gives us that $d(v_D(s^L) + 1) = d(v_D(s^L)) + \sigma_{r,k}(v_D(u_q) - dv_D(t))$. Thus, $d = \sigma_{r,k}(v_D(u_q) - dv_D(t))$. Since d divides

$\sigma_{r,k}$, we have $d = \sigma_{r,k}$ and $v_D(u_q) - dv_D(t) = 1$. Therefore, by (3.41), $q = 2$.

4. Characterization of Cylinderlike Surfaces with Multiple Fibrations

Theorem 4.1 *Let $T = Spec\ A$ be a cylinderlike surface with $T \not\cong \mathbf{A}^2$. Then T has multiple cylindrical fibrations if and only if there are $s, t \in A$; $\lambda \in k^*$; and $d, e \in \mathbf{Z}$ with $d > 1$, $0 < e < d$, and $gcd(e, d) = 1$ such that*

$$A = k\Big[s, t, \frac{t^d}{s^e}, \frac{t^{d-b}}{s^{e-a}}, \frac{t^b(t^d - \lambda s^e)}{s^{e+a}}, \frac{(t^d - \lambda s^e)^d}{s^{de+1}}\Big]$$

here $a, b \in \mathbf{Z}^+$ with $0 < b < d$ and $1 = ad - be$.

Proof. Suppose T has multiple cylindrical fibrations. Let $s, t \in A$; $R \in \mathbf{Z}^+$; and $\alpha_i, \beta_i \in k$ satisfy (1.1)-(1.3) and Lemma 1.3. By Proposition 2.1, $R = 1$. Assume $\alpha_1 = \beta_1 = 0$. From (1.2) and (1.3), we have $d \in \mathbf{Z}$ and a height one prime ideal P with $d > 1$ and $(s) = P^d$. Let D be the prime divisor corresponding to P and let $e = v_D(t)$. By (1.5) and Lemma 1.3, $0 < e < d$. Let $q \in \mathbf{Z}^+$, $n_i \in \mathbf{Z}^+$, and $u_i \in k[s,t]$ be as in (1.7)-(1.10). Then, by Proposition 3.1, $q = 2$. By (1.7), $n_1 = d$ and, by [C2, (2.1)-(2.4)], $n_1 = \frac{d}{\gcd(d,e)}$. Thus, $\gcd(d,e) = 1$. By [C2, (2.5)-(2.13)], in (1.9) we have $e_{0,1} = \frac{e}{\gcd(d,e)} = e$. Therefore, by (1.8) and (1.9), there is $\lambda \in k^*$ with $u_2 = t^d - \lambda s^e$ and, by Proposition 3.1, $v_D(t^d - \lambda s^e) = de + 1$. Let $a, b \in \mathbf{Z}$ with $0 \le b < d$ and $1 = ad - be$. Then $a > 0$ and, since $d > 1$, we have $b > 0$. By [C2, Th. 4.1], $A = k[s, t, v, v_1, v_2, w_2]$ where, by [C2; (3.12), (3.13), and (3.16)], $v_1 = \frac{t^d}{s^e}$, $v_2 = \frac{t^b(t^d - \lambda s^e)}{s^{e+a}}$, $w_2 = \frac{(t^d - \lambda s^e)^d}{s^{de+1}}$, and $v = \frac{t^{d-b}}{s^{e-a}}$.

Now let $s, t \in A$; $\lambda \in k^*$; and $d, e \in \mathbf{Z}^+$ be as above with $A = k[s, t, w, v, u, x]$ where $w = \frac{t^d}{s^e}$, $v = \frac{t^{d-b}}{s^{e-a}}$, $u = \frac{t^b(t^d - \lambda s^e)}{s^{e+a}}$, and $x = \frac{(t^d - \lambda s^e)^d}{s^{de+1}}$. By [C2, Th. 4.1], T is a cylinderlike surface and, by the proof of [C2, Prop. 3.1], the only height one prime ideal which contains s is $P = (s, t, w - \gamma, v, u^d - \xi x)$ for some $\gamma, \xi \in k^*$. We also have that $\{V((s - \alpha)) | \alpha \in k\}$ is a cylindrical fibration of T. Let $y = \frac{t(t^d - \lambda s^e)^{d-e}}{s^{de - e^2 + 1}}$ and

$$B = k\Big[x, y, \frac{y^d}{x^{d-e}}, \frac{y^b}{x^{b-a}}, \frac{y^{d-b}(y^d - \lambda x^{d-e})}{x^{a-b+2d-2e}}, \frac{(y^d - \lambda x^{d-e})^d}{x^{d^2 - de + 1}}\Big].$$

Since $y = \frac{u^{d-e}}{w^{b-a}}$ and since there are no maximal ideals containing both s and w, we have that y is regular on T and, therefore, $y \in A$. Similarly, $t \in B$. We also have $s = \frac{(y^d - \lambda x^{d-e})^d}{x^{d^2 - de + 1}}$, $w = \frac{y^d}{x^{d-e}}$, $v = \frac{y^{d-b}(y^d - \lambda x^{d-e})}{x^{a-b+2d-2e}}$, and $u = \frac{y^b}{x^{b-a}}$. Thus, $A = B$ and $\{V((x - \zeta)) | \zeta \in k\}$ gives us a second cylindrical fibration of T.

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