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Alexandre Grothendieck's EGA V

Part IV

(Interpretation and Rendition of his 'prenotes')

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§9 Change of Projective Embedding

9.1. For every integer $n > 0$ let $P(n) = P(Sym^n(\mathcal{E}))$, we have an evident immersion $u_n: P \to P(n)$, since $\mathcal{O}(n)$ is generated by its sections over every open affine of S and that $p_*(\mathcal{O}_P(n))Sym^n(\mathcal{E})$ [Illegible] where $p: P \to S$ is the projection. If $f: X \to P$ is an unramified morphism (resp. an immersion) it is the same with $u_n f: X \to P(n)$. There is sometimes an advantage in the study of X in replacing f by $u_n f$ in order to avoid a very special behavior of f sometimes embarrassing in certain respects. (An example of such peculiarity is the one indicated (sic) in 8.12 b), where $\mathcal{Y}^{\text{sing}} \to \check{P}$ has an image of dimension $r - 1$ but gives rise to an inseparable extension of fields. Another one is that given by the quadric surfaces in P^3 to know that all the singular hyperplane sections are geometrically reducible. (in spite of the fact that X is geometrically irreducible.)

Proposition 9.2. *We suppose $S = Spec(k)$, X smooth over k, and $f: X \to P$ unramified. Let $n \geq 2$ and let us consider $f_n = u_n f$. Then $f_n: X \to P(n)$ satisfies the equivalent conditions of 8.8, in particular for*

$\xi \in \check{P}(n)$ in the complement of a set of codimension ≥ 2, the corresponding hyperplane section Y_ξ is smooth or admits only a finite number of non-smooth points which are geometrically ordinary singularities. If f is an immersion there is at most one such singular point and it is rational over $k(\xi)$.

N.B. One would have to announce 8.8 in a manner such as not to exclude the case where f is not an immersion. The verification is essentially trivial under the condition (ii bis) of 8.8. For sure we should make explicit in 9.1 that the hyperplane sections of X relative to f_n are nothing else but the "sections" of X by hypersurfaces of degree n in place of hyperplanes.

Proposition 9.3. *Suppose that X is geometrically irreducible and that* $\dim f(X) \geq 2$*, let $x \in X(k)$ let $n \geq 2$.*

a) *Let us consider the linear family of hyperplane sections of X relative to $f_n = u_n f$ which pass through x, its generic element defines a $Y_\xi^{(n)}$ which is geometrically irreducible.*
b) *Let L be a linear subvariety of P passing through $f(x)$ and not containing $f(X)$. Let us consider the linear family of hyperplane sections of X relative to f_n defined by the hyperplanes of $P(n)$ "tangent to L at x" (i.e. defined by the n-forms over P which over L are zero of order at least two at x), its generic member is a $Y_\xi^{(n)}$ which is geometrically irreducible.*
c) *Let us suppose that X is smooth at x and that $n \geq 3$ where $f(X)$ is not contained in a plane defined over $\bar{k}$. Let us consider the family of hyperplane sections $Y_\xi^{(n)}$ of X relative to f_n which are "tangent to X at x". Then the generic member of the latter defines a $Y_\xi^{(n)}$ that is geometrically irreducible.*

The proof is essentially trivial in terms of the criteria of the end of the previous section. Taking an affine model of P containing $f(x)$ we are reduced in a) to finding three polynomials in the coordinates $T_1, T_2, \ldots, T_r$ of degree ≤ 2, let them be P, Q, and R such that $Q(t)/P(t)$ and $R(t)/P(t)$ are algebraically independent over k in K (where K is the function field of X and $t = (t_1, \ldots, t_r)$ is the system of elements of K defined by the T_i); in b) we require also that P, Q, and R should vanish at the order two at least on L which we can in addition suppose to be defined by the equations $T_1, \ldots, T_s = 0$; finally, in c) it is the same but L is the image of the tangent space to X at x and if necessary, we may take P, Q, and R of degree 3, i.e. more breathing space. The hypothesis that $\dim f(X) \geq 2$ signifies that the transcendence degree of $K(t_1, t_2, \ldots, t_r)$ over k is ≥ 2, i.e. we can find t_1, t_2 let us say algebraically independent. In a) we take therefore $P = T_1$, $Q = T_1^2$, $R = T_1 T_2$, in b) we do the same noting that we may there choose t_1 resulting from T_1 *zero over* L, due to the fact that $f(X) \not\subset L^*$ (which implies that there exists an index i between one and s

such that $t_s \neq 0$, so that t_s is not a constant (since t_s is zero at x) therefore t_s is not algebraic over k[1] (N.B. we may suppose k algebraically closed). The case c) follows from b) except in the case where $f(X)$ is contained in the image, L, by f of the tangent space to X at x. [If dim $X = 2$ this case is effectively exceptional (the trace of a quadric surface tangent to a plane on that plane is in general formed by two intersecting lines and is therefore not irreducible)]. But to treat that case, in the forms P, Q, and R made explicit above we may replace evidently X by L itself, where the solution is trivial. (If dim $L = 2$ take $P = T_r^2$, $Q = T_r^3$, $R = T_r^2 T_{r-1}$ and note that $Q/P = T_r$ and $R/P = T_{r-1}$ are linear forms independent over L, therefore algebraically independent. If dim $L \geq 3$ then T_{r-2}, T_{r-1}, T_r are linearly independent over L and we take

$$P = T_r^2,\ Q = T_{r-1}^2,\ R = T_{r-2}^2$$

Conjugating with 8.18 we find a Corollary 9.4. ([Interpr.] to be stated)

Finally we must combine the latter with 9.2 in order to find a recapitulating theorem in the "excellent case."

Theorem 9.5. *(If X is smooth, proper and geometrically integral over k and $f\colon X \to P$ is unramified, X of dimension ≥ 2 by considering the result [Illegible] when $X \to P$ is an immersion (unicity of the singular point of Y_ξ).)*

§10 Pencils of hyperplane sections and fibrations of blown up varieties

10.1. Let Z be the P-exceptional set in $\check{P}$ relative to a constructible property P such that Z is a constructible subset of $\check{P}$. Let us suppose $S = Spec(k)$. We will see (cf. No. 12 where we catch up with things which should have come in without any doubt in (previous Nos.) that in order to have codim$(Z, \check{P}) \geq 2$ it is necessary and sufficient that "every sufficiently general line" L in $\check{P}$ should not meet Z or even $\bar{Z}$ and it suffices that there should exist a single L in $\check{P}$ not meeting Z *[should be Z; AG's error]*. If k is infinite it is necessary and sufficient that there should exist a single straight line L in $\check{P}$ that does not meet $\bar{Z}$. We call linear pencil of hyperplane sections of "X" defined by the straight line L in $\check{P}$ the L-prescheme $\mathcal{Y}_L$ (definition is valid for any S). Then the previous thoughts together with results of Nos. 8 and 9 give us criteria for the existence of such pencils having the fibers $\mathcal{Y}_\xi$ $(\xi \in L)$ all satisfying the property P first of all in the case where S is an infinite base field. Taking into account 8.2, if for every prime cycle associated with X we have dim $f(T) > 0$ then we can (by taking the property $P' = P+$ condition of regularity for ϕ_ξ)

[1] [Interpr.] added by interpreter

require that the pencil Y_L should be flat over L. In the case where S is arbitrary we can again, by the procedure of 7.1, construct such a pencil over an open neighborhood of a given point s of S provided $k(s)$ is infinite and Z is closed (which is ensured in diverse various miscelaneous cases by the results of Par. 5 and the assumption that $X \to S$ is proper). To do it right it would be convenient after general explaination of this type to give recapitulating statements where we effectively apply the preceding results for a certain number of properties of this nature (and also comprising module properties). As a minimum in this sense we must give here the reformulation of 9.5 in terms of linear pencils – a fact constantly used in geometric applications.

10.2. By polarity, a straight line $\mathcal{L}$ in $\check{P}$ corresponds to linear subvariety L^0 of codim 2 in P (S arbitrary). Let us put $T = X \times_P L^0$. Another way to describe T is as follows: L is defined by a locally free quotient of rank 2 of $\check{\mathcal{E}}$ or what is the same defined by a submodule, locally a direct factor $\mathcal{F}$ of $\mathcal{E}$ everywhere of rank two. Let us consider the composition homomorphism

$$\mathcal{F}_X \longrightarrow \mathcal{E}_X \longrightarrow \mathcal{O}_X(1),$$

then T is nothing else but the scheme of zeros of this composite homomorphism or, what is the same, it is defined by the ideal J, image of the corresponding homomorphism (obtained by twisting by $\mathcal{O}_X(-1)$):

$$\mathcal{F}_X(-1) \longrightarrow \mathcal{O}_X.$$

Let us suppose that this homomorphism is regular which means that if we write down locally a totally ordered basis of $\mathcal{F}_X(-1)$ then its image in $\mathcal{O}_X$ forms an $\mathcal{O}_X$-regular sequence, a condition that does not depend on the basis chosen and that can be announced intrinsically also by saying that $\mathcal{F}_X(-1) \otimes \mathcal{O}_X/\mathcal{J} \to \mathcal{J}/\mathcal{J}^2$ is an isomorphism and that $V(\mathcal{J}) = T \to X$ is a regular immersion [NB: we should somewhere reveal the general situation with a homomorphism $\mathcal{G} \to \mathcal{O}_X$, $\mathcal{G}$ locally free over the prescheme X, for example in the section about regular immersions] we have then

Theorem 10.2. *With the above hypothesis the linear pencil $\mathcal{Y}_L$ with the canonical projection $\mathcal{Y}_L \to X$ is X isomorphic in a unique fashion to the blow-up of the prescheme X obtained with T.*

To understand the meaning of this theorem it would be convenient to notice at the beginning of the section or No. that if $S = \mathrm{Spec}(k)$ then for a 'sufficiently general' straight line L in $\check{P}$ the condition of regularity is verified (cf. *catching up indicated in No. 12* namely for 5.3.) In what follows in the construction of "good" linear pencils indicated or anticipated at the beginning of the present No., we could require that the described pencil should satisfy the said condition (which is a condition of the same type but different from the one that consists in requiring that for every

$\xi \in L$, ϕ_ξ should be $X_{k(\xi)}$-regular). We should include the condition in question in the proposed recapitulating statements.

On the other hand, practically 10.2 is used only in the situation of 9.5, which makes it desirable not to announce the reformulation of 9.5 in terms of pencils, until after 10.2, in order to be able to include in the statement in question also the isomorphism of the pencil with a blown up (i.e. to give a description of the situation permitting a suitable reference). We obtain thus a way for every projective smooth geometrically connected X of dim ≥ 2 over an infinite field k, to find a non-empty closed smooth subscheme of codimension two at every one of its points such that the blown up scheme admits a fibration over P^1, with all fibers being geometrically integral and such that all the fibers are smooth except at most a finite number, the latter having at most one geometrically singular point and such a point being rational over k and geometrically an ordinary singularity.

This explains the importance of a thorough study (just started now) of such fibrations with singular fibers to reduce in a certain measure (to some extent the study of projective smooth varieties of dimension d to those of families depending on one parameter) or projective varieties of dimension $(d-1)$ that may have ordinary singularities.

The statement 10.2 is a more or less immediate consequence of the following which is completely independent of the story of hyperplane sections and would be without a doubt better in its place in an extra paragraph on "regular immersions."

Proposition 10.3 is crossed out. Ask AG if that is his intention.

Proposition 10.3. *Let X be a prescheme, $\mathcal{G}$ a quasi-coherent module over X and $u\colon \mathcal{G} \to \mathcal{O}$ a homomorphism, $\mathcal{J} = u(G)$, $T = V(\mathcal{J})$. Let $\widetilde{X}$ be deduced from X by blowing up T. Let us consider on the other hand $p = p(G)$, the canonical homomorphism $G_p \to \mathcal{O}(1)$ and its kernel $\mathcal{H}$ (such that we have the exact sequence $0 \to \mathcal{H} \to \mathcal{G}_P \to \mathcal{O}_P(1) \to 0$) the homomorphism $u_P\colon \mathcal{G}_P \to \mathcal{O}_P$ and the quasi-coherent ideal $\mathcal{K} = u_P(\mathcal{H}) \to \mathcal{O}_P$. Then $\widetilde{X}$ is canonically isomorphic to a closed subscheme of $V(\mathcal{K})$. If G is locally free and u is "regular" then the above isomorphism is an isomorphism of $\widetilde{X}$ with $V(\mathcal{K})$ itself in this case $\mathcal{H}$ is locally free over P and $\mathcal{H} \to \mathcal{O}_P$ whose prescheme of zero is $\widetilde{X}$ is also regular.*

The first statement is almost trivial. The second one is an exercise which does not cause any difficulty (I have not done it in detail thinking that you can deduce it just as well as I).

If in 10.2, $S = Spec\, k$ and X is of dimension ≤ 1 then the assumption of regularity is equivalent to $T = \phi$ so that $\mathcal{Y}_L \to X$ is an isomorphism. We find therefore by conjugating with 9.2:

Corollary 10.4 (of 10.2). *Let X be a smooth curve geometrically connected in a projective space P over an infinite field k and let $n \geq 2$. Then there exists a linear pencil of n-forms over P defining a morphism $X \to P^1$*

having the following property: the morphism is generically étale of degree d and for every geometric point s of P^1, X_s *is étale over the algebraically closed field* $k' = k(s)$ *or it is* k' *isomorphic to the sum of* $d-2$ *schemes Spec* k' *and the scheme* $I'_k = Spec\, k'[t]/(t^2)$. *In the language of the forefathers: there is at most one point of ramification and it is "quadratic."*

§11 Grassmanians

Since we will now use linear subvarieties of P not only of relative dimension 0 and $n-1$ it is clear that we shall need some notions about grassmanians and some *sorites*] in the nature of 'elementary geometry' on the constructions concerning linear varieties which should all come at the beginning of the paragraph. In addition *one takes in practice sometimes* any linear sections and not only hyperplane sections and it is proper to review in this enlarged scope all the previous Nos. sections.

Let $\mathcal{E}$ be a quasi-coherent Module over the prescheme S, and let n be an integer > 0. Let us consider the functor $(\mathrm{Sch})^0/S \to (\mathrm{Ens})$ defined by $Grass_n(\mathcal{E})\ (S') =$ quotient Modules, locally free and of rank n of $\mathcal{E}_{s'}$.

This functor is representable and the prescheme over S which represents it will also be denoted $Grass_n(\mathcal{E})$. To prove the representability consider the natural functor homomorphism

$$Grass_n(\mathcal{E}) \to Grass_1\left(\bigwedge^n \mathcal{E}\right) = \mathcal{P}\left(\bigwedge^n \mathcal{E}\right)$$

defined by associating with every locally free quotient of rank n, $\mathcal{G}$ of $\mathcal{E}_{s'}$ the locally free Module of rank one $\bigwedge^n \mathcal{G}$ considered as a quotient of $\bigwedge^n \mathcal{E}_{s'}$ We prove as in *Seminaire Cartan* that this morphism if "representable by closed immersions" such that $Grass_n(\mathcal{E})$ appears as a closed subscheme of $P(\bigwedge^n \mathcal{E})$; in particular it is separated over S and quasi-compact over S and if $\mathcal{E}$ is of finite type it is projective over S. If $\mathcal{E}$ is of finite presentation then that is also the case for $Grass_n(\mathcal{E})$: indeed we may suppose that S is affine$S = \mathrm{Spec}(A)$ so that $\mathcal{E}$ comes from a module of finite type over a subring of finite type of A – since the formation of $Grass_n(\mathcal{E})$ is evidently compatible with base change over S.

Since $\mathcal{E}$ is locally free then $Grass_n(\mathcal{E})$ is *smooth* over S with geometrically connected fibers. This comes from a more precise fact: If $\mathcal{E}$ is free of rank r then $Grass_n(\mathcal{E})$ may be covered by $\binom{r}{n}$ open subsets each one of which is S isomorphic to affine space of relative dimension $n(r-n)$ over S. This decomposition corresponds to the choice, thanks to the base of $\mathcal{E}$ to $\binom{r}{n}$ decompositions of $\mathcal{E}$ by exact sequences

(s) $$O \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

with $\mathcal{E}'$ locally free of rank n. Such an exact sequence allows us to define a sub-functor $Grass_n(s)$ of $Grass_n(\mathcal{E})$ by limiting ourselves to quotients G of $\mathcal{E}'_s$ locally free of rank n, such that the composed homomorphism $\mathcal{E}'_{s'} \to \mathcal{E}'_s \to G$ should be surjective (therefore bijective). But the inclusion $Grass_n(S) \to Grass_n(\mathcal{E})$ is representable by open immersion and on the other hand $Grass_n(\mathcal{E})$ is representable by open immersion and on the other hand $Grass_n(S)$ is canonically isomorphic to the fiber bundle $V(Hom_{\mathcal{O}_s}(\mathcal{E}', \mathcal{E}''))$.

As a result, for example, of this particular structure we may mention that if $s \in S$, then ($\mathcal{E}$ being locally free of finite rank) every point of $Grass_n(\mathcal{E})$ with values in $k(s)$ lifts to a section over a neighborhood of s. On the other hand, if $S = \text{Spec}(k)$, k an infinite field, then every open non-empty subset of $Grass_n(\mathcal{E})$ contains a k-rational point. A point of $Grass_n(\mathcal{E})$ with values in S, i.e. a locally free quotient Module $\mathcal{G}$ of rank n of $\mathcal{E}$ canonically defines a subscheme of $P(\mathcal{E})$, (i.e. to say) $P(G)$. Such a subscheme (but without *imposing* anything on the rank of G) is called a *linear subvariety* of $P(\mathcal{E})$ (relative to S if there is a possibility of confusion). It is therefore a projective fibration of relative dimension $(n-1)$ if $n \geq 1$, (and empty if $n = 0$). We immediately verify that the section of $Grass_n(\mathcal{E})$, i.e. G is known if we know the linear subvariety corresponding to $P(\mathcal{E})$. In this manner the grassmanian can be interpreted as representing the functor ("linear subvarieties of relative dimension $n-1$ of $P_{s'}$") for S' variable in $n \geq 1$. It is furthermore possible to give an intrinsic characterization of the latter functor, i.e. of the notion of linear subvariety of relative dimension m they are closed subschemes of P, smooth over S and everywhere of relative dimension m and that are of "projective degree one" at every $s \in S$; this characterization will be given in a later chapter and we shall not need it at all here.

Let us again suppose that $\mathcal{E}$ is locally free of rank r, let $\check{\mathcal{E}}$ be its dual. Then by a polarity we find a canonical isomorphism $Grass_n(\mathcal{E}) \simeq Grass_{r-n}(\check{\mathcal{E}})$ that assigns to a quotient $\mathcal{G}$ of $\mathcal{E}$ the quotient $\check{\mathcal{E}}/\check{\mathcal{G}}$ of $\check{\mathcal{E}}$. From the point of view of linear varieties to a linear variety L of relative dimension m of P there corresponds the linear dual variety L^0 of relative dimension $(r-1) + -1 - m$ of $\check{P}$, i.e. of relative codimension $(m+1)$ in $\check{P}$. (N.B. $(r-1)$ is here the relative common dimension of P and $\check{P}$ over S), which we may visualize geometrically *as follows*. Let us first of all take $n = r-1$, we find an isomorphism $P(\check{\mathcal{E}}) \simeq Grass_{r-1}(\mathcal{E})$ that allows us to identify the points of $\check{P}$ with values in S (let us say) as linear subvarieties of codim 1 of P (called *again hyperplanes* of P).

To tell it short, L^0 consists of hyperplanes which *contain* the linear subvariety L of P (by which of course we mean that the points of L with values in S' are the hyperplanes in $P_{S'}$ that contain $L_{S'}$). This follows from the fact (that should have occurred at the same time as the fact that a linear subvariety L of P determines a locally free quotient $\mathcal{G}$ of $\mathcal{E}$ from which it comes that if $\mathcal{G}$ and g' are two locally free quotients of E (not necessarily of the same rank) then $P(\mathcal{G}') \subset P(\mathcal{G})$ (as the linear subvarieties of $P(\mathcal{E})$), if $\mathcal{G}'$ is majorized by $\mathcal{G}$ (and the inclusion $P(\mathcal{G}') \to P(\mathcal{G})$ is nothing else but the deduced morphism from $\mathcal{G} \to \mathcal{G}'$).

Here is a minimum of sorites which we must have at our disposal. The complete list cannot in any case be fixed until the other Nos. of the present paragraph are written up.

It seems to me convenient to introduce also the functor

$$Grass(\mathcal{E})(S') = \text{set of quotient modules locally free (of rank not specified) of } \mathcal{E}_{S'}$$

[illegible] then $Grass(\mathcal{E})$ is representable by $\coprod_{n \geq 0} Grass_n(\mathcal{E})$. The linear subvarieties of $P(\mathcal{E})$ are indeed defined by sections of $Grass(\mathcal{E})$ over S (NB: the rank, i.e. the relative dimension may vary if S is not connected).

§12 Generalization of the previous results to linear sections

Complements to notations. If $P = P(\mathcal{E})$, $\mathcal{E}$ any quasi-coherent Module, we set also $Grass_n(P) = Grass_{n+1}(\mathcal{E})$ so that $Grass_n(P)$ corresponds to linear subvarieties of *dimension* n in P; this is valid for $n \geq -1$ if we agree that dim $= -1$ means *empty*. If $\mathcal{E}$ is locally free it would be advisable to introduce

$$Grass^n(P) = Grass_{n-1}(\check{P}) = Grass_n(\check{\mathcal{E}})$$

which corresponds to linear subvarieties of codimension n in P. If $\mathcal{E}$ is of rank $r+1$ [illegible] P of relative dimension r, we have a canonical isomoprhism $Grass^n(P) = Grass_{r-n}(P)$. In what follows, we suppose $\mathcal{E}$ fixed locally free of rank r and we are interested in linear subvarieties of P of given codimension m, thus in $\mathrm{Gr}^m = \mathrm{Gr}^m(P) = \mathrm{Gr}_m(\check{\mathcal{E}})$.

Over that prescheme we have therefore a canonical quotient $\mathcal{G}$ locally free of rank m of $\mathcal{E}_{\mathrm{Gr}}$, let us call it F. The natural incidence prescheme over $P \times_S \mathrm{Gr}^m$, which represents the subfunctor of the product functor corresponds to the couples consisting of a section of $P_{S'}$ and a linear subvariety of codimension m of $P_{S'}$ *containing* the latter, it can be made explicit therefore in the following way: let $T = P \times_S \mathrm{Gr}^m$ (or if we prefer any prescheme relative or over this product), then over T we have $\mathcal{E}_T$ the quotient $\mathcal{O}_T(1)$ and the sub-Module locally direct factor of $\check{\mathcal{G}}_T$. We consider the composition of the canonical homomorphisms $\check{\mathcal{G}}_T \to \mathcal{O}_T(1)$ which by transposition corresponds also to the analogous composite homomorphism of the sub-Module $\mathcal{O}_T(-1)$ of $\check{\mathcal{E}}_T$ into the quotient $\mathcal{G}_T$:

$$\mathcal{O}_T(-1) \to \mathcal{G}_T$$

and may also be considered as defined by a section of $\mathcal{G}_T(1), \phi^m \in \Gamma(T, \mathcal{G}_T(1))$. The incidence prescheme (resp. its inverse image in T) is nothing else but the prescheme of zeros of the one or the other homomorphism or else of the section ϕ^m. We could denote the incidence prescheme by $\mathcal{H}^{(m)}$ for $m = 1$; we recover the one from No. 1. If X is over P we may set $\mathcal{Y}^{(m)} = X \times_P \mathcal{H}^{(m)}$ and define by this the notation $\mathcal{Y}^m$ if ξ is a point of Gr^m with values in an S'. Therefore the Y_ξ^m are "*linear sections*" of X over P (or rather of X'_S over P'_S by linear subvarieties of codimension m of P or rather of P'_S.)

I use this opportunity for a notational self-criticism which could come since already No. 1. This is the fact to make the arbitrary correspondence in order to indicate an object Y that corresponds to X the letter Y to X (so that if X becomes Z we no longer know very well what to take.) This inconvenience has already led me into some incoherent notations a few times.

Perhaps in the more general context with an integer m as here suggests a reasonable solution: to write $\mathcal{X}^{(m)}$ instead of $\mathcal{Y}^{(m)}$, thus $\mathcal{X}^{(1)}$ in place of $\mathcal{Y}$ in No. 1. In such a way we might approximately have $\mathcal{X}^{(m)(m')} = \mathcal{X}^{m+m'}$. I am going to try such notation in what follows. Evidently even the exponent is open to criticism since it is current practice in algebraic geometry to denote by an exponent the dimension of the varieties which enter into play. But since we shall never make use of this type of convention, I think that we have a free hand as far as that matter is concerned.

We see immediately that, in the preceding construction of $\mathcal{X}^{(m)}$, we have a canonical P-isomorphism $\mathcal{X}^{(m)} = Grass_m(\check{\mathcal{F}})$ where $\mathcal{F}_X \cong f^*(\Omega^1_{P/S})(1)$ is the kernel of $\mathcal{E}_X \to \mathcal{O}_X(1)$, in particular $\mathcal{X}^{(m)}$ is smooth over X with geometrically integral fibers. (In fact, rational varieties of dimension $m(r-m)$.) Of course, the verification reduces to the case $X = P$ and because of this it belongs just as the previous considerations to the generalities on grassmanians (which I am sure you are going to "magnify" in a separate paragraph).

We now have a perfect counterpart of the diagram from No. 1. Again a forgotten point: as a prescheme over Gr^m, $\mathcal{H}^{(m)}$ is canonically isomorphic to $P(\mathcal{E}_{\mathrm{Gr}}/\check{\mathcal{G}})$; it is therefore an excellent projective fibration (but of course we may not conclude this in general for $\mathcal{X}^{(m)}$ over Gr^m).

The Proposition 2.1 can be transposed without change. In Proposition 2.2 it should read: it is necessary and sufficient that for every $x \in Z$ we have dim $\bar{x} \leq m-1$. For the proof we may, for example, restrict ourselves to 2.6 by considering a generic linear variety of codimension m, as the intersection of m independent generic hyperplanes. Dieudonne demerdetur (Interpr. – joke – "démerde toi" with Latin flavor.)

From the writing up point of view if (as it seems preferable to me) we make from the start m general, it seems preferable to prove 2.6 at the same time, where, of course, dim $X - 1$ is replaced by dim $X - m$ (and by implying that the dimension < 0 in the formula means that the considered set is empty).

Corollary 2.3 is read by replacing 'finite' by "of dimension $\leq m-1$." Corollary 2.4 similar. The same for 2.5, replace dim $f(X_i) > 0$

by dim $f(X_i) \geq m$ and the same change in 2.7.

The Proposition 2.8 remains true as stated in 2.9 replace finite by dim $\leq m-1$. The same for 2.10, 2.11.

The statement 2.12 remains valid as such with a proof essentially unchanged (compare also further down comments to No. 8); 2.13 replace finite by dim $\leq m-1$. Theorem 2.14 stays valid as such, 2.15 by replacing finite by dim $\leq m-1$. 2.16 is valid as stated in 2.17 replace finite by dim $\leq m-1$. 2.18 as it is.

For 3.1, we can state it for any m, supposing that dim $f(X) \geq m+1$, but I propose to keep the principal statement in the case of a hypersurface and to give the general case as a corollary as a remark (it can be deduced immediately by the usual procedure of taking independent generic hyperplanes).

At least it would be amusing to make explicit (state) a generalized version of Lemma 3.1.1.... For (3.2) read dim $f(X) \geq m+1$ in 3.3 replace dim $f(X_i) \geq 2$ by dim $f(X_i) \geq m+1$ and in the definition of $\mathfrak{S}$, dim $f(Z) = 0$ by dim $f(Z) \leq m-1$.

The general considerations of No. 4 apply as such to the case of any m. The same is true about 4.2 and 4.3 by replacing in b) (v) and (vi) the dimension condition by dim $f(X) \geq m+1$. Analogous change in 4.4 b)

The "laius" of 5.1 goes as such. In 5.2 it is necessary to remember that ϕ becomes a section $\phi_\xi^{(m)}$ of $\mathcal{G}_T(1)$ (where $T = X_S \times \mathrm{Gr}^m$) inducing the sections $\phi_\xi^{(m)}$ of

$$\mathcal{O}_{X_k}(1) \otimes_{k(\xi)} \mathcal{G}(\xi) \qquad\qquad (\text{for } \xi \in \mathrm{Gr}^m).$$

But in general we shall explain in par. 19 that if we have a section ϕ of a locally free Module of rank m over a prescheme this means that such a section is $\mathcal{F}$-regular for a given Module $\mathcal{F}$ in terms of a local basis, this means that we have an $\mathcal{F}$-regular sequence of m sections of $\mathcal{O}_X$ (and it will be necessary to verify that this is independent of the chosen basis). In the case $m = 1$ we have the intrinsic evident interpretation mentioned in 5.2. With this language convention 5.3 remains valid as such, as 5.4.

The first part of Remark 5.5 admits a generalization to the case of m arbitrary: If F_S is (S_m) then the condition of regularity mentioned for ϕ_ξ can be expressed in a purely dimensional manner.

The second part of Remark 5.5 is valid as such for any m. Theorem 5.6 extends as such, so does 5.7.

Proposition 6.1. *Read* dim $f(X_i) \geq m+1$ *and later* dim $f(Z) \leq m-1$.

The general "laius" of 7.1 are valid as such in the case of any m. 7.2, 7.3 mutatis mutandis [Latin joke] (pay attention in 7.2 to the notation m, confusing everywhere), on the other hand in the proof of 7.4 we no longer need to proceed by successive approximations but we may take rightaway a linear section of codimension $m = n$.

In 8.2 replace condition dim $Y_\xi >$ dim X by dim $Y_\xi >$ dim $X - m$ and the hypothesis dim $f(X_i) > 0$ by dim $f(X_i) \geq m$. Similar modification is in the sequel to 8.2. Since 8.3 gives an example there is no point in changing it so we keep $m = 1$.

I leave it as an exercise to you, that should be done with care, to find good statements for any m corresponding to 8.4, 8.5, and 8.6 (pages 30, 31, 32). It is not necessary to do this exercise unless you feel like doing it, but do not get bored!

I think that essentially all the developments of No. 8 except 8.6 can be adapted to the case of linear sections with any m. To do it formally would be without any doubt quite a long and fastidious exercise. I have to admit that I do not know any applications depending essentially on the analysis of this more general situation, so we are not really obliged to include these developments in these Elements. On the other hand, experience proves that the fact of writing up in this more general context forces often to better unscrew and understand better the whole caboudle and often with no cost, in addition a certain number of syntax exercises in an exclusively geometric context like here will do no harm and of course it is not at all excluded that one day we will use it or need it and we will be happy to find it ready to use. Still I leave up to you the whole decision about this subject and I summarize the statements that we could perhaps give along this connection.

Let us again assume that $f: X \to P$ is unramified and that X is smooth over S with components of dimension $\geq m$. Then $\mathfrak{X}^{(m)} = V^0$, we distinguish therefore the sub-prescheme $\mathfrak{X}^{(m)\mathrm{sing}} = V^1$ of the singular zeros of ϕ^m relative to Grass^m, which is also formed geometrically from pairs (x, ξ) such that the linear variety L_ξ cuts by excess the tangent space to X at x (considered as linear subvarieties of P), i.e. such that the two spaces do not generate all of P. In opposition with what happens for $m = 1$, if m is arbitrary the morphism $\mathfrak{X}^{(m)\mathrm{sing}} \to X$ is not in general smooth since the variety of L which pass through x and cut by excess a given linear subvariety $T \ni x$, is not in general smooth over k: this variety is only the closure of the smooth subvariety formed by ξ such that the dimension of $T \cap L_\xi$ is just **one** more than the "normal" dimension $(n - m)$ ($n = \dim T$, $m = \operatorname{codim} L$). Except an error, the set (contained in the relative supersingular set) V'' introduced in par. 16 (complements) is nothing else but the set formed by the couples (x, L_ξ) such that the dimension of $T_x \cap L_\xi$ is $\geq n - m + 2$ so that $V' - V''$ is smooth over S and except an error it is exactly the same as the set of smooth points of V' over X. (The verification of this point requires a study of the filtration of the Grassman scheme according to the dimensions of intersection of L variables and T fixed, except an error we find that the following notch of the filtration (when we define the filtration either set theoretic or schematic way), the filtration is formed exactly of the non-smooth points of the using the lemma from page 16 of the complements to par. 16. This study would form therefore one of the No. of a "geometric" paragraph devoted to grassmanians.)

If we also define $V^{(k)}$ as the sub-scheme of $X^{(m)}$ corresponding to $\dim T_x \cap L \geq n - m + k$, we find by an immediate calculation that

$$\dim Grass^m(P) - \dim V^{(k)} = (k-1)(n-m) + k^2$$

at least for the reasonable restrictions $k \leq m$, $k \leq r - m$, up to an error of calculation. (NB this follows more generally from a calculation of the dimensions of the "cells" which intervene in the filtration of grassmanians, which were alluded to above).

For $k = 2$, we find a difference of dimension ≥ 4, so that the image of V'' in Grass [illegible] is of codimension ≥ 4 so that if we are interested in what happens outside of subsets of the Grass of codimension ≥ 2 we may forget V''.

On the other hand, in $X_{Grass^m} - V''$ over $Grass^m$ the situation is that of the good case anticipated in the complements to par. 16. Relative to the base scheme S: $V' - V''$ is indeed smooth over S (being such over X) of relative dimension equal to **one** less than that of $Grass^m$ over S (as we see by putting $k = 1$ in the above formula). Thus the results of the loc cit [Latin] apply, in particular we find the fact that the set of supersingular points of ϕ^m relative to $Grass^m$ is nothing else but $V'' \cup V^2$ where V^2 is the sub-prescheme of ramification of $V' - V'' \to Grass^m$. We may therefore say that outside of V'' the supersingular zeros result from collapsing of (at least) two ordinary singular zeros. (but we do not have to say this).

In this way we have essentially the equivalent of 8.7 a) and b). It should be possible to give an equivalent condition for 8.7 c) by using the explicit description of the tangent bundle to $Grass^m$ (analogous to the case $m = 1$), it implies also that for a geometric point of $V' - V''$, unramified over $Grass^m$, to know its image in $Grass^m$ implies knowing its image in P as long as the first image is a smooth point of the closed image of V' in $Grass^m$ (we assume S is the spectrum of a field). I could give a more precise statement upon request.

Once we grant this, we have the evident corollaries generalizing 8.8, 8.9, 8.11. It is without a doubt also possible to formulate in the case of an arbitrary m the other propositions of paragraph 8.

If this demands additional efforts of writing up, we could give up this generalization, even if we include the previous differential developments.

The same is true about the results of No. 9.

As for No. 10, the situation studied there generalizes to the case of any m in the following manner. We fix a linear subvariety C of P of codimension $(m+1)$ and we consider the projective space Q of linear subvarieties L of P of codimension m passing through C. Q is a closed subscheme of $Grass^m$, in particular we could construct $\mathfrak{X}_Q^{(m)}$ which we propose to study.

A first point, which has to be in any case to figure in the text is that $\mathfrak{X}_Q^{(m)} \to X$ is again birational at least if C cuts "regularly" X and precisely

$\mathcal{X}_Q^{(m)}$ is in this case canonically isomorphic to the prescheme deduced from X by blowing up $X \times_P C$: the proof of this fact is nothing else but 10.2, via 10.3. A second point which is of some interest but which we do not absolutely have to include consists in saying that if we choose C "fairly general", then $\mathcal{X}_Q^{(m)} \to Q$ has certain pleasant properties, the most classical one being this: X being assumed smooth over $S = Spec(k)$ and of dim $n \geq m$ proper and geometrically irreducible, then, for 'fairly general' C the set T of $\xi \in Q$ for which $\mathcal{X}^{(m)}$ is not smooth of dimension $(n-m)$ over $k(\xi)$, is geometrically irreducible over $k(\xi)$ and of codimension one in Q and the set T' of $\xi \in T$ for which $\mathcal{X}^{(m)}$ is "supersingular" at least at one point is rare (nowhere dense) in T; finally, if $F: X \to P$ is an immersion, then after extending T' a little, for every $\xi \in T - T'$ there is exactly one non-smooth point in $X_\xi^{(m)}$ and the latter is rational over $k(\xi)$. I forgot to specify in the statement that we assume $X \to P$ unramified and that we have to initially replace f by $\phi_n f$, $n \geq 2$ (where ϕ_n is defined in 9.1). The most natural way of proving this statement seems to be to use the subscheme Z (denoted T in 8.8) of $Grass^m$ such that $\mathcal{X}^{(m)}$ is "singular": we see that, under the given conditions, it is geometrically irreducible of codimension one and that the subscheme Z^1 corresponding to $\mathcal{X}^{(m)}$ "supersingular" is nowhere dense.

It remains, therefore, to prove a lemma of the following nature: let Z be a closed subset of $Grass^m$ of codimension q then defining $Q(C)$ in terms of C as above for every C "fairly general" the intersection $Q(C) \cap Z$ is of codimension $\geq q$ in $Q(C)$; moreover if Z is geometrically irreducible so is $Q(C) \cap Z$ if Z is "fairly general."