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Alexandre Grothendieck's EGA V
Part V
Elementary Morphisms and M. Artin's Theorem
(Interpretation and Rendition of his 'prenotes')

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§13 Elementary morphisms and the Theorem of M. Artin

Definition 13.1. A morphism $f\colon X \to Y$ of prescheme is called an "elementary morphism" if X is Y-isomorphic to a prescheme of the form $X' - Z$ where $X' \to Y$ is a smooth projective morphism with geometrically connected fibers of dimension one, and where Z is closed sub-prescheme of X' such that the morphism $Z \to Y$ is étale surjective and of constant degree. A morphism is called *polyelementary* if it is a composition of elementary morphisms. A prescheme X over a field k is called *polyelementary* (over k) if the structural morphism $X \to \mathrm{Spec}(k)$ is poly-elementary.[1]

[1] cf. p. 117 Milne 'Étale Cohomology' and SGA IV

Theorem 13.2 (M. Artin).

Let X be a geometrically irreducible prescheme over a field k, perfect and infinite, x a smooth point of X, then x admits a fundamental system of open polyelementary neighborhoods.

Replacing X by a given neighborhood of x, it is enough to prove that there exists an open elementary neighborhood of x in X.

Arguing by induction on the dimension n of X, we are reduced to proving that if $n > 0$ then there exists an open neighborhood U of x and an elementary morphism $f: U \to V$, V being a smooth scheme over k (necessarily geometrically irreducible and of dimension $n-1$). The case $n = 0$ is trivial since X is then k-isomorphic to $\mathrm{Spec}(k)$ which is polyelementary over $\mathrm{Spec}(k)$ (in 13.1 we do not exclude the composition of the empty family of morphisms). We should mention it in one way or another in 13.1. The necessity to assume that k is first of all perfect appears already in the case $n = 1$ where we take for X' the projective normal canonical model of the function field K of X (cf. Chap II, Par. 7)[2] the fact that k is perfect insures that X' is smooth over k (since X' is regular anyway) and it also insures that $Z = X' - X$ with the induced reduced structure is étale over k. Let us now treat the general case where we can finally assume $n \geq 2$.

We may obviously suppose that X is affine, therefore quasi-projective. Then by replacing X by a projective closure we may assume that X is projective under reservation to prove that *every* neighborhood contains an open neighborhood U having an *elementary* morphism $U \to V$. Also, replacing X by its normalization (finite over X, therefore projective),[3] which does not change it in the neighborhood of x, we may assume that X is normal; therefore, k being perfect, geometrically normal over k.

The benefit of this hypothesis is that the set Z of points of X, where X is not smooth over k is of codim ≥ 2. Let us choose a projective immersion $i: X \to P^r$, as we obtain a fundamental system of neighborhoods of x in P^r by taking the sections not vanishing at x of the various $\mathcal{O}_P^{(n)}$, $n > 0$, and we conclude that every neighborhood of x contains a neighborhood of the form $X - Y$, where Y is a closed subset of X containing Z, purely of dimension $(n-1)$ and such tht $x \notin Y$.

We give Y the induced reduced structure such that (k being perfect) the singular set of Y is of dimension $\leq n-2$. By enlarging the previous set Z we find a closed subset $Z \subset Y$ of dimension $\leq n-2$ containing the geometrically singular set of X *and* of Y.

The idea of the proof is to fiber X by its intersections with linear subvarieties L of P of codimension $(n-1)$ containing a given linear subvariety C of codimension n. To this end we will need the following:

Lemma 13.3. *With the preceding notations for X, Y, Z, ($X \supset Y \supset Z$) closed subschemes of P_k^r of dimension n, $n-1$ and $\leq n-2$, $X - Z$, $Y - Z$*

[2] EGA II, 7.4 [Interpr.]

[3] EGA II, [Interpr.]

smooth, Z of dimension $\leq n-2$ if needed (if k is of characteristic $p>0$) by replacing the projective immersion $i\colon X \to P^r$ by any "multiple" ϕ_n^i $(n \geq 2)$ as in No. 9, there exists a linear subvariety L_0 of P^r of codimension $(n-1)$ and having the following virtues (nice properties)

a) $L_0 \cap Z = \phi = \emptyset$

b) $L_0 \cap X$ *is smooth of dimension 1*

c) $L_0 \cap Y$ *is smooth of dimension 0.*

(N.B. k denotes an infinite field *without the necessity of being perfect here.*)

Let us assume this lemma and let us show how we can deduce the existence of an open neighborhood U of x contained in $X-Y$ and having an elementary morphism $U \to V$. There exists a linear subvariety C of L_0 of codimension n in P^r, i.e. of codimension 1 in L_0 not meting the finite set $(L_0 \cap Y) \cup \{x\}$. Let $T = X \cap C$ so that T is a subscheme of X étale over k, non-empty and not contianing x and *disjoint* from Y. Let us consider on the other hand the subscheme Q of $\text{Grass}^{n-1}(p)$ corresponding to linear subvarieties of P^r containing C such that Q is a projective space of dimension $(n-1)$; in particular it is smooth over k and of dimension $(n-1)$. Then L_0 corresponds to a point ξ_0 of $Q(k)$. Let us consider, on the other hand, (with the general notations introduced before) the inverse image $\mathcal{X}_Q^{n-1}$ of $\mathcal{X}^{n-1}$ by the immersion $Q \to \text{Grass}^{n-1}$ and also the inverse images $\mathcal{Y}_Q^{n-1}$ and $\mathcal{T}_Q^{n-1}$ which are also closed disjoint subschemes of $\mathcal{X}_Q^{n-1}$, let p, q, r be the structural projections of these schemes to Q. Then by essentially assumption p is smooth *at the points lying over* ξ_0, and q is étale at the points *lying over* ξ_0; this is also the case for r as we see that $\mathcal{T}_Q^{n-1}$ is nothing else but $\mathcal{T} \times_k Q$ (Q isomorhism). Finally the morphism p is proper, and taking into account that X is geometrically connected, the fibers of p are geometrically connected (Bertini's theorem). Consequently, there exists an open neighborhood V of Q in X such that $\mathcal{X}_Q^{n-1} \mid V = X'$ is proper and smooth over V with geometrically connected fibers, and since the fiber of ξ_0 is nothing else but $X \cap L_0$, it is of dimension 1, we may suppose that the fibers of X' over V are all of dimension 1. Finally, taking V sufficiently small, we may suppose that $\mathcal{Y}_Q^{n-1} \mid V$ and $\mathcal{T}_Q^{n-1} \mid V$ are étale over V so that the *sum* prescheme of Z' of this two (which may be identified with a closed prescheme of X') is étale over V. Consequently, putting $U = X' - Z'$, the morphism $U \to V$ is an elementary morphism. But U is also an open subset of $\mathcal{X}'' = \mathcal{X}_Q^{n-1} - \mathcal{Y}_Q^{n-1} - \mathcal{Z}_Q^{n-1}$, the inverse image of $X - Y - T$ in $\mathcal{X}_Q^{n-1}$; on the other hand, $X'' \to X - Y - T$ is obviously an isomorphism (since $\mathcal{X}_Q^{n-1} - \mathcal{T}_Q^{n-1} \to X - T$ is an isomorphism). Therefore U is identified to an open subset of $X - Y - T$, an open subset containing, furthermore, $L_0 \cap X$ and a fortiori x. This is the desired neighborhood of x contained in $X - Y$.

It remains only to prove Lemma 13.3. As usual, it suffices to prove that the generic linear subvariety of codimension $(n-1)$ passing through x called L has properties a), b), c). To prove a) as well as the dimensional content of b) and c), this follows immediately from 2.3 (reviewed and corrected in No. 12) applied (as in a reasoning already done in No. 8) to the projective space of straight lines passing through x and the image of Z in that space by a conic projection from x. [It might be useful in addition to make explicit certain results obtained by this method concerning the linear sections by linear subvarieties subject to the condition of passing through a fixed linear subvariety. In the text or in a separate No.] For the smoothness in b) and c) we can because of a) replace X and Y respectively by $X - Z$ and $Y - Z$ which are smooth and we are reduced to proving this: Let $f\colon X \to P$ be an unramified morphism with X smooth over k and let $X \in Pk)$ such that x does not belong to the image of any component of X of dimension $< m$ (*irreducible component*?) then if η is the generic point of the subgrassmanian of $\mathrm{Grass}^m P$ formed from linear varieties L of codimension m passing through x, $\mathcal{X}^{(m)}$ is smooth over k at least if k is of characteristic zero and the opposite case, by replacing f by $\phi_n f$, n an integer ≥ 2.

This is a remorse to No. 9, which itself follows from the remorse following No. 8: With the notations of 8.8 (supposing that X is irreducible, which is acceptable for the problem that we are discussing) if we have $\operatorname{codim} T \geq 2$ or if $\mathcal{Y}^{\text{sing}} \to T$ is generically étale (condition that is automatically satisfied if k is of characteristic zero or on condition of replacing f by $\phi_n f$ with $n \geq 2$, cf No. 9, then for the hyperplane H_η passing through a generic x $\mathcal{X}_\eta^{(1)}$ is smooth of dimension $(n-1)$, except in the case where we have $f(X) = \{x\}$ (thus $n = 0$). This result accepted which liquidates evidently the special case $m = 1$ of our remorse, we obtain immediately the case of a general m, by induction on m by noticing that up to a change of basis $\mathcal{X}_\eta^{(m)}$ is obtained by taking an L' of codimension $(m-1)$ passing through x, L' and H being generic independent for these properties (i.e. in orthodox terms we place ourselves at the generic point of the scheme of pairs (L', H) and by taking the linear section of X by $L' \cap H$: we may start by taking its section by H, which is smooth by inductive assumption.) This type of reasoning already used to generalize 2.6, for example to linear sections of any codimension m deserves to be made explicit one good time in general so that we may refer to it without entering every time into the details, a bit heavy of a complete presentation.

It remains to prove the corollary announced of 8.8 in the case $m = 1$. If $\operatorname{codim} T \geq 2$, since on the other hand the hyperplane Q of $\check{P}$ of H, such that $x \in H_\xi$ is of codimension 1, its generic point η cannot be an element of T and we win. [Fr]. (we are done?) In the case $\operatorname{codim} T = 1$, since T is irreducible, we cannot have $\eta \in T$ unless $Q = T$, i.e. in geometric terms (supposing k algebraically closed which is acceptable for every $z \in X$ the tangent space to X at z (or rather its image by f'_z passes through x. Let us prove that this cannot happen unless $\mathcal{X}^{\text{sing}} \to T$ is generically étale, i.e. if we are under the condition of 8.8, except in the case $f(x) = \{x\}$ thus X of dimension zero. Indeed 8.7 c) (which expresses essentially the symmetry in the relation between X and its "dual" T) implies that for almost every point $z \in X(k)$, $f(z)$ is orthogonal to the tangent space to T at a certain point, therefore (since $T = Q$) orthogonal to Q, where $f(z) = x$ hence $f(X) = \{x\}$. This puts an end to the proof of our remorses and thus of 13.2. N.B. The reasoning does not work if we replace x by a linear subvariety X of dim > 0, and if we subject H to passing through C; indeed, there is no reason to suppose (taking for example dim $C = r-2$ the greatest possible for which H can still effectively vary) that T contains the straight line C^0 taking for example a non-singular quadric in P in any characteristic. But it is possible that such phenomena cannot happen anymore for $\phi_n f$, $n \geq 2$; we could pose the questions as a remark in No. 9.

Remark 13.4.

a) We have already observed that the hypothesis that k should be perfect is essential for the validity of 13.2. On the contrary, it is plausible that the hypothesis k is infinite is never redundant. We did not try an ad hoc reasoning for the case where k is finite and we only note that in this case the application of 13.2 to the algebraic closure of k and usual arguments show that we may find a finite extension k' of k such that for the point of $X_{k'}$ over x there exist open polyelementary neighborhoods relative to k'.

b) If in 13.2 we abandon the hypothesis that X is geometrically irreducible, the conclusion obviously does not remain the same (since an algebraic poly-elementary scheme is geometrically irreducible!). The conclusion holds, however, in a weaker form which is obtained by omitting in definition 13.1 the word "connected" as this is shown by the proof that we have given.

c) (To be possibly included in the statement of 13.2) With the notations of 13.2, let Φ be a finite subset of X of smooth points of X and let us suppose that Φ is contained in an affine open subset of X. Then Φ possesses a fundamental system of open polyelementary neighborhoods. Evidently we may suppose that Φ consists of closed points. The proof is essentially the same except that we specify 13.3 in slightly different form: there exists a linear subvariety C of P^r of codimension n, not meeting $\Phi \cup Y$ and such that for every $x_i \in \Phi(\bar{k})$, the linear subvariety of L_i of codimension $(n-1)$ generated by C at x_i has the properties a), b), c) of 13.3. To verify this point we note that it suffices to verify that the *generic* C has the above-mentioned properties since for such a variety each L_i is generic among the L of codimension n passing through x_i so that we can apply 13.3 in the initial form (or at least in the form that we have proven which was [Fr]: every L_0 "sufficiently general" passing through x has properties a), b), c).

d) By proceeding as explained in 7.1, we may give variants of 13.2 in the case where we replace the base field k by a general base Y prescheme. Let us remark the following (without proof): Let $f\colon X \to Y$ be a flat projective morphism with geometrically irreducible and (R_2) fibers, S' a subscheme of X finite over S, $x \in S$, suppose that for every $x \in S'$ over s, X_s is smooth over $k(s)$ at x. Then there exists an open neighborhood U of s and an open neighborhood V of $S' \mid U$ in $X \mid U$ such that $V \to U$ is poly-elementary. If Y is a closed subscheme of X not meeting S' and such that the set Z of points where Y is not smooth over S verifies $\dim Z_s \leq \dim X_s - 2$, then we may above take V contained in $X - Y$.

e) One of the reasons why 13.2 is interesting is the topological structure particularly simple of the elementary algebraic schemes U. For example if the base field is the field of complex numbers and if U^{an} denotes the analytic space associated to U then the homotopy groups $\pi_i(U^{\mathrm{an}})$ are zero for $i \neq 1$ and π_1 is a successive extension of free groups. Thus U^{an} is a "space $K(\pi_1, 1)$", *classifying for* π_1, more precisely its universal covering space is homeomorphic to $\mathbb{C}^n$ and a fortiori is contractible and this covering is a "universal principal fibration" with group π_1.