Open Problems

Electronic Scottish Café

Problem #1
Paul Erdős
Geometry

Anning and I (Bull. Amer. Math. Soc. 1945) proved that if $X_1, X_2, \ldots$ is an infinite set of points in the plane and all distances $d(X_1, X_2)$ are integers, then the points are on a line. When I told this theorem to Stan he asked: Let $X_1, X_2 \ldots$ be a dense set in the plane. Prove that not all the distances $d(X_1, X_2)$ can be rational. This is probably difficult and is still open. Perhaps, in fact, there is a dense subset where all distances between pairs of points of the subset are irrational.

Problem #2
Paul Erdős

Let $X_1, X_2, \ldots, X_n$ be n points in the plane. Assume that if two distances $d(X_1, X_j)$ differ, they differ by at least one. Is it then true that the diameter $D(X_1, X_2, \ldots, X_n)$ is $> cn$? And perhaps for $n > n_0$ $D(X_1, \ldots, X_n) \geq n-1$. $n-1$ is of course best possible (if true) if the points are $0, 1, \ldots, n-1$. In three dimensional space

$$D(X_1, \ldots, X_n) > cn$$

is certainly not true. Perhaps $D(X_1, \ldots, X_n) > cn^{2/3}$ holds.

Assume next that all the distances $d(X_1, X_j)$ are different and

$$(d(X_i, X_j) \cdot d(X_k, X_\ell)) \geq 1$$

i.e., two distances differ by at least one. Then of course the diameter $D(X_1, \ldots, X_n)$ is $\geq \binom{n}{2}$. I think that perhaps in any dimension

$$D(X_1, \ldots, X_n) > (1 + \sigma(1))n^2$$

I can only prove this if the points are on a line, i.e., in dimension one.

Problem #3
Paul Erdős
Number Theory

Let $P(n)$ be the greatest prime factor of n. Is it true that the density of integers with $P(n) > P(n+1)$ is $\frac{1}{2}$? The answer is no doubt affirmative, but this seems very hard. An old result of mine states that the density of integers n for which $d(n+1) > d(n)$ is $\frac{1}{2}$, where $d(n)$ is the number of divisors of n.

Problem #4
Paul Erdős

Is it true that for every k you can find k consecutive integers satisfying $P(n) < P(n+1) < \cdots < P(n+k-1)$? Again, the answer surely is yes, but I cannot prove it. I offer \$100.00 for a proof. It is easy to prove that there are k consecutive integers for which $d(n) < d(n+1) < \cdots < d(n+k-1)$.

Problem #5
Paul Erdős

Old problem of mine: Is it true that if $a_1 < a_2 < \ldots$ is an infinite sequence of integers for which

$$\sum \frac{1}{a_i} = \infty \tag{1}$$

then for eery k the a's contain an arithmetic progression of k terms? (\$3,000.00 for a proof or disproof.) If true, then for every k there are k primes which form an arithmetic progression. It is not even known that (1) implies that there are three a's in an arithmetic progression, i.e., that $a_1 + a_j = 2a_k$ is solvable.

Problem #6

Peter D. Lax
Courant Institute of Mathematical Sciences
New York University

Show or disprove that, if S is a compact set in a locally convex topological space, then its closed convex hull is compact.