Ulam Quarterly — Volume 2, Number 1, 1993

Wavelets Generated by Three—Term Two–Scale Relations

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Abstract

We show there exists a continuous compactly supported function φ defined on the reals satsifying

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1) + c_2\varphi(2x-2)$$

if and only if $c_1 = 1$, $c_0 + c_2 = 1$, and $|c_0|, |c_2| < 1$. The functions φ are examples of scaling functions from the theory of wavelets.

§1. Introduction

In the last decade, wavelets have emerged as an exciting new development in mathematics which has significant applications in many areas of science and engineering. Their development may be viewed as the culmination of work from the areas of harmonic analysis, physics, and signal processing in engineering. This has resulted in a virtual explosion of interest in wavelet analysis among scientists from many different areas.

As is well–known, the tool of multiresolution analysis, originally developed from ideas in image analysis (cf. [3]), provides a simple method of constructing wavelets from scaling functions satisfying a two–scale relation of the form

$$\varphi(x) = \sum_{k=-\infty}^{\infty} c_k\varphi(2x-k)$$

which satisfy the partition of unity property

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = \text{constant } \neq 0.$$

The two–scale relation generating the Meyer scaling function, contains infinitely many non–zero terms. However, the classical Haar scaling function, the B–spline scaling functions, and the Daubechies scaling functions are determined by symbols having finitely many non–zero terms, a property desirable for computations. Here we consider the scaling functions generated by two–scale relations of length three.

As stated in [1], the Haar scaling function is the unique solution of the two–term two–scale relation

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1).$$

However, it seems to be thought that the only continuous (on the entire real line) solution of the three–term two–scale relation

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1) + c_2\varphi(2x-2)$$

is the tent function, for which $c_0 = c_2 = \frac{1}{2}$, and $c_1 = 1$. The next section contains a proof of

1.1 Main Theorem. *There exists a continuous, compactly supported function φ, satisfying*

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1) + c_2\varphi(2x-2)$$

if and only if

$$c_1 = 1; \quad c_0 + c_2 = 1; \quad 0 < |c_0|, \ |c_2| < 1.$$

Moreover, φ is Hölder continuous with exponent $-\log_2 \max(|c_0|, |c_2|)$.

It is interesting to note that the coefficients in the scaling relation are allowed to be complex numbers. In section three we discuss some additional related results.

§2. Continuous three term scaling functions

We begin by establishing the necessity of the conditions in Theorem 1.1, even without the partition of unity hypothesis.

2.1 Lemma. *Let complex numbers c_0, c_1, c_2 be given. Suppose φ is a continuous compactly supported function φ which is not identically zero. If φ satisfies*

$$\varphi(x) = c_0\varphi(2x) + c_1\varphi(2x-1) + c_2\varphi(2x-2), \tag{1}$$

then the support of φ is a subset of the interval $[0, 2]$, $c_1 = 1$, $\varphi(1) \neq 0$, $c_0 + c_2 = 1$, and $|c_0|, |c_2| < 1$.

Proof. Let a denote the left end point of the support of φ. If $a < 0$, then for all ϵ small enough $2(a + \epsilon) < a$. In particular, both $2(a + \epsilon) - 2$ and

$2(a+\epsilon)-1$ are strictly less than a. With $x=a+\epsilon$, (1) implies $\varphi(x)=0$. This contadicts the definition of a; and therefore $a \geq 0$. Similarly, the right end point of the support of φ is at most 2.

To show $\varphi(1) \neq 0$ we argue by contradiction. Accordingly, suppose $\varphi(0)=0$. We will show, by induction on n, that $\varphi(j2^{-n})=0$ for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Since the support of φ is the interval $[0,2]$ and $\varphi(0)=0$, this is true for $n=0$. If $\varphi(j2^{-n})=0$ for all $j \in \mathbb{Z}$ (with n fixed), then, using the induction hypothesis,

$$\begin{aligned}\varphi(\tfrac{j}{2^{n+1}}) =& c_0\varphi(\tfrac{j}{2^n}) + c_1\varphi(\tfrac{j-2^n}{2^n}) + c_2\varphi(\tfrac{j-2^{n-1}}{2^n}) \\ =& 0.\end{aligned}$$

Thus, φ is zero on the dyadic rationals and therefore, by continuity, identically zero. This contradiction implies $\varphi(1) \neq 0$. Now, with $x=1$, the relation (1) gives $\varphi(1)=c_1\varphi(1)$. Whence $c_1=1$.

Using induction, it is straightforward to show that

$$\begin{aligned}\varphi(\frac{1}{2^n}) &= c_2^n \quad \text{and} \\ \varphi(2-\frac{1}{2^n}) &= c_0^n.\end{aligned} \tag{2}$$

Consequently, since φ is continuous at 0 and 2 and since $\varphi(0)=\varphi(2)=0$, $|c_0|, |c_2| < 1$.

Using (2) and induction, it is possible to verify the formula

$$\varphi(1+\frac{1}{2^n}) = c_0[1+c_2+\ldots c_2^{n-1}].$$

From (2) and the continuity of φ at 0, it follows that $c_0\sum_{m=0}^{\infty} c_2^m = 1$; i.e., $c_0+c_2=1$. This completes the proof. □

The *tent* function T, defined by

$$T(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2-x & \text{for } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

is an example of a function satisfying (1) with $c_0=c_2=\frac{1}{2}$ and $c_1=1$. A remark in [1], page 129, suggests that there are no other continuous solutions of (1) with compact support.

Fix a complex number α such that both $|\alpha|$ and $|1-\alpha|$ are strictly less than 1. In the remainder of this section, we construct a continuous function supported in $[0,2]$ such that

$$\varphi(x) = \alpha\varphi(2x) + \varphi(2x-1) + (1-\alpha)\varphi(2x-2). \tag{3}$$

The first step is to construct a function φ on the dyadic rationals which satisfies (3) on the dyadic rationals.

For $n \in \mathbb{N}$, let

$$A_n = \left\{ \frac{j}{2^n} : j \in \mathbb{Z} \right\}$$

and let A denote the dyadic rationals, viewed as the union of the A_n. φ is defined by recursion on the sets A_n. Define F_0 on A_0 by

$$F_0(j) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j \neq 1. \end{cases}$$

Assuming F_n has been defined on A_n, define F_{n+1} on A_{n+1} by

$$F_{n+1}(\tfrac{j}{2^{n+1}}) = \alpha F_n(\tfrac{j}{2^n}) + F_n(\tfrac{j-2^n}{2^n}) + (1-\alpha)F_n(\tfrac{j-2^{n+1}}{2^n}). \tag{4}$$

It is straightforward to verify, arguing by induction, that $F(j2^{-n}) = 0$ if $j2^{-n} \notin (0,2)$. Moreover, we claim that F_{n+1} extends F_n; i.e., F_{n+1} restricted to A_n is F_n. This is immediate when $n = 0$. Now suppose it is true for n. We have, using, in order, the definition of F_{n+2}, the induction hypothesis, and the definition of F_{n+1},

$$\begin{aligned} F_{n+2}(\tfrac{j}{2^{n+1}}) =& F(\tfrac{2j}{2^{n+2}}) \\ =& \alpha F_{n+1}(\tfrac{j}{2^n}) + F_{n+1}(\tfrac{j-2^n}{2^n}) + (1-\alpha)F_{n+1}(\tfrac{j-2^{n+1}}{2^n}) \\ =& \alpha F_n(\tfrac{j}{2^n}) + F_n(\tfrac{j-2^n}{2^n}) + (1-\alpha)F_n(\tfrac{j-2^{n+1}}{2^n}) \\ =& F_{n+1}(\tfrac{j}{2^{n+1}}). \end{aligned}$$

This shows that the function φ on A given by $\varphi(j2^{-n}) = F_n(j2^{-n})$ is well defined and satisfies the scaling relation (3).

We will show that φ is uniformly continuous, in fact Hölder continuous on A, and hence extends to be continuous on the reals. The following lemma expresses a crucial symmetry condition satisfied by φ, namely that on the dyadic rationals φ satisfies the partition of unity property.

2.2 Lemma. *The function φ has the following important property. For each natural number n and for each integer j such that $2^n \leq j \leq 2^{n+1} - 1$,*

$$\varphi(\tfrac{j}{2^n}) - \varphi(\tfrac{j+1}{2^n}) = \varphi(\tfrac{j+1}{2^n} - 1) - \varphi(\tfrac{j}{2^n} - 1). \tag{5}$$

Proof. We argue by induction on n. The case $n = 0$ is easy to verify. Suppose (5) holds for a fixed n and let $2^{n+1} \leq j \leq 2^{n+2} - 1$ be given. From the scaling relation (3) we have

$$\begin{aligned} \varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}}) = (1-\alpha)[\varphi(\tfrac{j}{2^n} - 2) - \varphi(\tfrac{j+1}{2^n} - 2)] \\ + [\varphi(\tfrac{j}{2^n} - 1) - \varphi(\tfrac{j+1}{2^n} - 1)] \end{aligned} \tag{6}$$

We consider two cases. In the case $2^{n+1} \leq j \leq 3(2^n) - 1$, the induction hypothesis, applied to the last term of (6) and to $j - 2^n$, obtains

$$\varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}}) = \alpha[\varphi(\tfrac{j+1}{2^n} - 2) - \varphi(\tfrac{j}{2^n} - 2)]. \tag{7}$$

On the other hand, using the scaling relation (3), the support of φ is in the interval $[0, 2]$, and $\frac{j+1}{2^{n+1}} - 1 \leq \frac{1}{2}$,

$$\varphi(\tfrac{j+1}{2^{n+1}} - 1) - \varphi(\tfrac{j}{2^{n+1}} - 1) = \alpha(\varphi(\tfrac{j+1}{2^n} - 2) - \varphi(\tfrac{j}{2^n} - 2)). \tag{8}$$

Comparing (7) and (8) shows that (5) is valid for $n+1$ provided $2^{n+1} \leq j \leq 3(2^n) - 1$. In the case $3(2^n) \leq j \leq 2^{n+2} - 1$, the scaling relation (3) gives

$$\varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}}) = (1-\alpha)[\varphi(\tfrac{j}{2^n} - 2) - \varphi(\tfrac{j+1}{2^n} - 2)], \tag{9}$$

since, in this case, $\frac{j}{2^{n+1}} \geq \frac{3}{2}$. On the other hand, again from (3),

$$\begin{aligned}\varphi(\tfrac{j}{2^{n+1}} - 1) - \varphi(\tfrac{j+1}{2^{n+1}} - 1) = \alpha[\varphi(\tfrac{j}{2^n} - 2) - \varphi(\tfrac{j+1}{2^n} - 2)]\\ + [\varphi(\tfrac{j}{2^n} - 3) - \varphi(\tfrac{j+1}{2^n} - 3)].\end{aligned} \tag{10}$$

Now, from the induction hypothesis, the last term in (10) is given by

$$\varphi(\tfrac{j+1}{2^n} - 2) - \varphi(\tfrac{j}{2^n} - 2).$$

Therefore, (10) becomes,

$$\varphi(\tfrac{j}{2^{n+1}} - 1) - \varphi(\tfrac{j+1}{2^{n+1}} - 1) = (1-\alpha)[\varphi(\tfrac{j+1}{2^n} - 2) - \varphi(\tfrac{j}{2^n} - 2)]. \tag{11}$$

Comparing (11) and (9) finishes the proof of the lemma. □

The following theorem expresses the Hölder continuity of φ on the dyadic rationals.

2.3 Theorem. *Define γ by $2^{-\gamma} = \max\{|\alpha|, |1-\alpha|\}$. For $n \in \mathbb{N}$ and $2^n \leq j \leq j+k \leq 2^{n+1}$,*

$$|\varphi(\tfrac{j}{2^n}) - \varphi(\tfrac{j+k}{2^n})| \leq (2^\gamma - 1)^{-1} (\tfrac{k}{2^n})^\gamma. \tag{12}$$

In particular, φ extends to the reals as a Hölder continuous function with exponent γ which, by continuity, satisfies the scaling relation (3).

Proof. We argue, as usual, by induction on n to first establish the Theorem when $k = 1$. The case $n = 0$ is straightforward to verify. Accordingly we assume (12) holds for a fixed n, $k = 1$, and $2^n \leq j \leq 2^{n+1} - 1$, and let j such that $2^{n+1} \leq j \leq 2^{n+2} - 1$ be given. By the scaling relation (3)

$$\begin{aligned}\varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}}) = [\varphi(\tfrac{j}{2^n} - 1) - \varphi(\tfrac{j+1}{2^n} - 1)]\\ + (1-\alpha)[\varphi(\tfrac{j}{2^n} - 2) - \varphi(\tfrac{j+1}{2^n} - 2)].\end{aligned} \tag{13}$$

If $2^{n+1} \le j \le 3(2^n) - 1$, then an application of Lemma 2.2 to the second term in (13) obtains

$$|\varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}})| = |\alpha||\varphi(\tfrac{j}{2^n} - 1) - \varphi(\tfrac{j+1}{2^n} - 1)| \tag{14}$$

If $3(2^n) \le j \le 2^{n+1} - 1$, then the first term on the right hand side of (13) vanishes and we obtain,

$$|\varphi(\tfrac{j}{2^{n+1}}) - \varphi(\tfrac{j+1}{2^{n+1}})| = |1 - \alpha||\varphi(\tfrac{j}{2^n} - 2) - \varphi(\tfrac{j+1}{2^n} - 2)| \tag{15}$$

With $C = \max\{|\alpha|, |1 - \alpha|\}$ and

$$\Delta_n = \max\{|\varphi(\tfrac{j}{2^n}) - \varphi(\tfrac{j+1}{2^n})| : 2^n \le j \le 2^{n+1} - 1\},$$

(14) and (15) imply

$$\Delta_{n+1} \le C\Delta_n. \tag{16}$$

Thus, $\Delta_n \le C^n$ for all natural numbers n.

By a dyadic interval, we mean an interval of the form $(\frac{j}{2^n}, \frac{j+1}{2^n}]$. Given natural numbers n, j, k such that $2^n \le j < j + k \le 2^{n+1}$, let J_0 denote the interval $J_0 = (\frac{j}{2^n}, \frac{j+k}{2^n}]$. Let J_1 denote the largest dyadic interval contained in J_0. Let J_2 denote the largest dyadic interval disjoint from J_1 and contained in J_0. In this way we obtain dyadic intervals $\{J_m\}_{m=1}^{\ell}$ which we denote

$$J_m = (\frac{j_m}{2^{n_m}}, \frac{j_m + 1}{2^{n_m}}],$$

such that

$$J_0 = \cup J_m,$$

and

$$\frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + ... + \frac{1}{2^{n_\ell}} = \frac{k}{2^n}. \tag{17}$$

We have

$$\varphi(\tfrac{j}{2^n}) - \varphi(\tfrac{j+k}{2^n}) = \sum_{m=1}^{\ell} \varphi(\tfrac{j_m}{2^{n_m}}) - \varphi(\tfrac{j_m+1}{2^{n_m}}). \tag{18}$$

Applying (16) to (18) obtains

$$\begin{aligned} |\varphi(\tfrac{j}{2^n}) - \varphi(\tfrac{j+k}{2^n})| &\le \sum_{m=1}^{\ell} (\frac{1}{2^{n_m}})^{\gamma} \\ &\le (\frac{1}{2^{n_1}})^{\gamma}(2^{\gamma} - 1)^{-1} \\ &\le (2^{\gamma} - 1)^{-1}(\tfrac{k}{2^n})^{\gamma}, \end{aligned}$$

since $\frac{1}{2^{n_1}} \le \frac{k}{2^n}$. The theorem follows. □

§3. Additional Results

If φ has $L^2(\mathbb{R})$ norm 1 and satisfies and the scaling relation (3) for some α complex such that both $|\alpha|$ and $|1-\alpha|$ are strictly less than 1, then it follows from the general theory (and can be verified directly) that

$$\sum \varphi(x-k) = \varphi(1)$$

for all x. Moreover, it is straightforward to compute the $L^2(\mathbb{R})$ inner products of translates of φ as

$$< \varphi(x-m), \varphi(x-n) >= \begin{cases} 1 & \text{if } \ m=n \\ \delta & \text{if } \ m=n-1 \\ \bar{\delta} & \text{if } \ m=n+1 \\ 0 & \text{otherwise} \end{cases}$$

where $\delta = \frac{\alpha(1-\bar{\alpha})}{(1-(\alpha-\bar{\alpha}))}$. Define

$$g(\theta) = \bar{\delta}e^{-i\theta} + 1 + \delta e^{i\theta}.$$

The set $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for the closure of their span if and only if g is a strictly positive function. Equivalently, if and only if $|\delta| < \frac{1}{2}$. This is easily seen to be the case under our assumption that $|\alpha| < 1$ and $|1-\alpha| < 1$. In particular, if φ is a continuous three–term scaling function with compact support, then the translates of φ automatically have the Riesz basis property.

By computing a Wiener–Hopf factorization of the function g, we can determine from φ a scaling function $\varphi^\sharp$ whose translates, $\varphi^\sharp(x-n)$, are mutually orthogonal. As a general procedure, there exists a function $q(\theta) = \sum_{n\,\geq\,0} q_n e^{in\theta}$ such that $g = |q|^2$. When it makes sense to write

$$\frac{1}{q}(\theta) = \sum_{n\,\geq\,0} h_n e^{in\theta},$$

$$\varphi^\sharp(x) = \sum_{n\,\geq\,0} h_n \varphi(x-n).$$

In our case,

$$q(\theta) = a + be^{i\theta}.$$

Further, in this case, $h_n = a(-b/a)^n$.

If α is real, then φ is increasing for $x \leq 1$ and decreasing for $x \geq 1$. In particular, the Hausdorff dimension of the graph of φ is 1. If α is not real the real and imaginary parts of φ are not piecewise monotonic functions. The Hausdorff dimension of these graphs may hold some surprise.

References

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