Ulam Quarterly — Volume 2, Number 2, 1993

Free Products of Matrices

Roger C. Alperin
Department of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192

Abstract

In this note we give some new examples of matrix groups which are free products. We use this in our study of the Burau representation in dimension four.

0. Introduction

In this note we give some new examples of matrices which generate their free product. The technique employed is based on similar methods developed by Humphries [H] and earlier by Bachmuth-Mochizuki [B-M]. Other methods have been used by Vinberg [V] in the case of reflections. Our interest in these questions was motivated by work of Dyer, Formanek and Grossman [D-F-G], which essentially reduces the question of faithfulness of the Burau representation of the braid group, $\mathcal{B}_4$, in dimension 4 to a problem about three reflections. More precisely, let

$$R = \begin{pmatrix} -1 & -(u^2+u^{-1}) & -(u+u^{-2}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proposition 0. *The Burau representation of $\mathcal{B}_4$ is faithful iff the matrices R, VRV^{-1} and $V^{-1}RV$ generate their free product, $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$, for some transcendental u.*

Proof. As shown in [D-F-G] the Burau representation is faithful iff the the matrices V and A generate their free product $\Gamma = \mathbf{Z}_3 * \mathbf{Z}_2$. Certainly if

R and V generate their free product then the subgroup Γ_0 generated by R, VRV^{-1} and $V^{-1}RV$ is the unique normal subgroup of index three in Γ; it is therefore generated by these three given matrices and is their free product. On the other hand if the subgroup Γ_0 is their free product then the group generated by R and V contains this group as a subgroup of index 1 or 3. However since Γ_0 has no elements of order three, $V \notin \Gamma_0$, and the index is three and therefore Γ is the free product. $\square$

§1. Some Free Products of Matrices

Consider the standard column space $\mathbb{C}^n$ with basis $e_1, e_2 \ldots, e_n$ and dual basis $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$. Let $A = (a_{ij})$ be an $n \times n$ matrix of complex numbers with $a_{ii} = 0$, $i = 1, 2, \ldots, n$. Let $D_i = \mathrm{diag}(d_{i1}, d_{i2}, \ldots, d_{in})$ be an invertible diagonal matrix. Let $\alpha_i = \hat{e}_i A$ denote the linear functionals $\sum_{j=1}^{n} a_{ij}\hat{e}_j$ and consider the transformations $R_i = D_i + e_i\alpha_i$ and its inverse $R_i^{-1} = D_i^{-1} + e_i\beta_i$, $\beta_i = -\alpha_i d_{ii}^{-1} D_i^{-1}$, for $i = 1, 2, \ldots, n$. The action of R_i, R_i^{-1} on row vectors is given by

$$(x_1, x_2, \ldots, x_n)R_i = (\ldots, x_j d_{ij} + x_i a_{ij}, \ldots, x_i d_{ii}, \ldots),$$
$$(x_1, x_2, \ldots, x_n)R_i^{-1} = (\ldots, x_j d_{ij}^{-1} - x_i a_{ij} d_{ii}^{-1} d_{ij}^{-1}, \ldots, x_i d_{ii}^{-1}, \ldots).$$

Using the simple recurrence relation $\delta_{ij(k+1)} = \delta_{ijk} d_{ij} + d_{ii}^k$ we have for all $k \geq 0$.

$$(x_1, x_2, \ldots, x_n)R_i^k = (\ldots, x_j d_{ij}^k + x_i a_{ij}\delta_{ijk}, \ldots, x_i d_{ii}^k, \ldots)$$
$$(x_1, x_2, \ldots, x_n)R_i^{-k} = (\ldots, x_j d_{ij}^{-k} - x_i a_{ij} d_{ii}^{-1} d_{ij}^{-1}\delta_{ij(-k)}, \ldots, x_i d_{ii}^{-k} \ldots,).$$

and $\delta_{ijk} = \frac{d_{ii}^k - d_{ij}^k}{d_{ii}^\varepsilon - d_{ij}^\varepsilon}$, $\varepsilon = \mathrm{sgn}(k)$, if $d_{ii} \neq d_{ij}$.

Notice also that $d_{ii}^{k-1} d_{ij}^{k-1}\delta_{ij(-k)} = \delta_{ij(k+1)}$. We shall abbreviate this notation in case i, j are understood by

$$\delta_k = \begin{cases} \delta_{ijk} & \mathrm{sgn}(k) > 0 \\ -d_{ii}^{-1} d_{ij}^{-1}\delta_{ij(-k)} & \mathrm{sgn}(k) < 0 \end{cases}$$

We shall consider the action of the group Γ generated by R_1, $R_2, \ldots, R_n$ on row vectors. Let $\mathcal{P}_{ij} = \{(x_1, x_2, \ldots, x_n) \mid |x_i a_{ij}| \geq 2|x_j|\}$ for $i \neq j$ and $\mathcal{S}_i = \bigcap_{r \neq i} \mathcal{P}_{rs}$. We shall assume the following conditions:

(0) $|d_{ij}| = 1 \quad \forall\, i, j.$

(1) $|\delta_{ijk}| \geq 1 \quad i \neq j, k$ whenever $R_i^k \neq I$

(2) $|a_{ij}a_{ji}| \geq 4 \quad i \neq j$

(3) $|a_{ij}a_{jk}| \geq 6|a_{ik}| \quad i, j, k$ distinct.

Lemma 1. *For all i, j, k, if $R_i^k \neq I$ then $\hat{e}_i \notin \mathcal{S}_j$ and $\hat{e}_i R_i^k \in \mathcal{S}_i$, hence $\mathcal{S}_i$ is nonempty for $n \geq 2$.*

Proof. Certainly $\hat{e}_i \notin \mathcal{P}_{ji}$, $j \neq i$ so $\hat{e}_i \notin \mathcal{S}_j$ for all i, j. Now consider the condition $\mathcal{P}_{rs}$ for $r \neq i$ for the vector $\hat{e}_i R_i^k$. In this case $(x_r d_{ir}^k + x_i a_{ir} \delta_k) a_{rs} = a_{ir} a_{rs} \delta_k$ and for $s \neq i$, $x_s d_{is}^k + x_i a_{is} \delta_k = a_{is} \delta_k$ while for $s = i$, $x_s d_{is}^k + x_i a_{is} \delta_k = d_{ii}^k$ so that the $\mathcal{P}_{rs}$ conditions are verified by

$$\begin{aligned} |a_{ir} a_{rs} \delta_k| &\geq 6 |a_{is} \delta_k| > 2 |a_{is} \delta_k|, && s \neq i \\ |a_{ir} a_{rs} \delta_k| &\geq 4 |\delta_k| > 2 |d_{ii}^k| && s = i \qquad \square \end{aligned}$$

Lemma 2. *For all i, k if $R_i^k \neq I$ and $i \neq j$ then $\mathcal{S}_j R_i^k \subset \mathcal{S}_i$*

Proof. We verify the conditions $\mathcal{P}_{rs}$, $r \neq i$ for a vector $x R_i^k$, $x \in \mathcal{S}_j$. Consider first the case $s = i$.

$$\begin{aligned} |(x_r d_{ir}^k + x_i a_{ir} \delta_k) a_{ri}| &\geq |a_{ri}| (|x_i a_{ir} \delta_k| - |x_r|) \\ &\geq |a_{ri}| \left(|x_i a_{ir} \delta_k| - \frac{|x_i a_{ir}|}{2} \right) \quad \text{since } x \in \mathcal{P}_{ir} \\ &\geq |a_{ri} a_{ir} x_i| \left(|\delta_k| - \frac{1}{2} \right) \\ &\geq 4 |x_i| \frac{1}{2} = 2 |d_{ii}^k x_i|. \end{aligned}$$

We assume now that $s \neq i$.

$$\begin{aligned} |(x_r d_{ir}^k + x_i a_{ir} \delta_k) a_{rs}| &\geq |a_{rs}| (|x_i a_{ir} \delta_k| - |x_r|) \\ &\geq |a_{ir} a_{rs} x_i| \left(|\delta_k| - \frac{|x_r|}{|x_i a_{ir}|} \right) \\ &\geq 6 |a_{is} x_i| \left(|\delta_k| - \frac{1}{2} \right) \quad \text{since} \quad x \in \mathcal{P}_{ir} \\ &\geq 2 |a_{is} x_i| \left(|\delta_k| + \frac{1}{2} \right) \quad \text{since} \quad |\delta_k| \geq 1 \\ &\geq 2 |a_{is} x_i| \left(|\delta_k| + \frac{|x_s|}{|x_i a_{is}|} \right) \quad \text{since} \quad x \in \mathcal{P}_{is} \\ &\geq 2 \left(|x_i a_{is} \delta_k| + |x_s| \right) \\ &\geq 2 \left(|x_i a_{is} \delta_k| + |x_s d_{is}^k| \right) \quad \text{since} \quad |d_{is}| = 1 \\ &\geq 2 |x_s d_{is}^k + x_i a_{is} \delta_k|. \end{aligned}$$

Thus the condition $\mathcal{P}_{rs}$ is satisfied. $\square$

Theorem 1. *If conditions (0)-(3) are satisfied then the transformations $R_1, R_2, \ldots, R_n$ generate their free product.*

Proof. We may assume that $n \geq 2$ and consider an alternating word $R = R_{i_1}^{k_1} R_{i_2}^{k_2} \cdots R_{i_m}^{k_m}$, $R_{i_j} \neq R_{i_{j+1}}$, $j = 1, \ldots, m-1$, $i_1 \neq i_m$. It follows from the previous Lemma 1 and Lemma 2 that $\hat{e}_{i_1} R \in \mathcal{S}_{i_m}$ so that $\hat{e}_{i_1} R \neq \hat{e}_{i_1}$. □

The conditions (0) and (1) are not easily satisfied but one case of interest are the complex reflections where $d_{ii} = \varepsilon_i = e^{2\pi I/l_i}$, a l_ith root of unity, and $d_{ij} = 1$, $i \neq j$. In this case, $|\delta_{ijk}| = \frac{|\varepsilon_i^k - 1|}{|\varepsilon_i - 1|} = \frac{\sin\left(\frac{\pi k}{l_i}\right)}{\sin\left(\frac{\pi}{l_i}\right)} \geq 1$, for $k \geq 1$, since $\sin(x)$ is increasing on $[0, \frac{\pi}{2})$.

Corollary 1. *The group generated by the complex reflections* $R_i = I - e_i\alpha_i$, *is their free product where* $\alpha_i = \sum_{j=1}^n a_{ij}\hat{e}_j$, $i = 1, 2, \ldots, n$, $\alpha_i(e_i) = 1 - e^{2\pi I/l_i}$, $l_i \geq 2$, *and* $|a_{ij}a_{ji}| \geq 4$, $i \neq j$, $|a_{ij}a_{jk}| \geq 6|a_{ik}|$, i, j, k *distinct.*

□

Using this theorem and its corollary, one may obtain other free products of matrix groups by dint of the subgroup structure theorems of free products. One can also use the technique of the theorem to show that these groups are discrete subgroups of $GL_n(\mathbb{C})$.

§2 Reflections

In this section we shall assume that each of $R_i = I - e_i\alpha_i$, $\alpha_i = \sum_{j=1}^n a_{ij}\hat{e}_j$, has the property that $\alpha_i(e_i) = 2$. The matrix $C = (\alpha_i(e_j))$ is called the Cartan matrix; Vinberg [V] has shown that if (v1) $a_{ij} < 0$ and (v2) $a_{ij}a_{ji} \geq 4$ $i \neq j$ then R_i generate their free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$. This result is a parallel to our Theorem 1.

We show that now that if C is Hermitian it is the matrix of a form invariant by the action of the R_i, $i = 1, 2, \ldots, n$. Let $\alpha_i^* = \sum_{j=1}^n \overline{a}_{ij} e_j$ be the vectors dual to α_i. Consider the transformation $T_i = e_i\alpha_i$ $1 \leq i \leq n$. Notice that $\sum_{i=1}^n T_i = \sum_{i=1}^n e_i\alpha_i = \sum_{i=1}^n e_i \sum_{j=1}^n a_{ij}\hat{e}_j = \sum_{i,j} a_{ij}e_i\hat{e}_j = C$ and $C^* = \sum_{i=1}^n T_i^* = \sum_{i=1}^n \alpha_i^* e_i = \sum \overline{a}_{ij} e_j \hat{e}_i$.

Proposition 1. *If* $C = C^*$ *then* $R_k^* C R_k = C$ *for* $k = 1, 2, \ldots, n$.

Proof. We first compute: $T_k^* T_i = \alpha_k^* \hat{e}_k e_i \alpha_i = \delta_{ki}\alpha_k^*\alpha_i$, $T_i T_k = e_i\alpha_i e_k\alpha_k = \alpha_i(e_k)e_i\alpha_k$. Therefore $T_k^* C = \sum_{i=1}^n T_k^* T_i = \alpha_k^*\alpha_k$ and $CT_k = \sum_{i=1}^n T_i T_k = \sum_{i=1}^n \alpha_i(e_k)e_i\alpha_k = \left(\sum_{i=1}^n \alpha_i(e_k)e_i\right)\alpha_k = \left(\sum_{i=1}^n a_{ik}e_i\right)\alpha_k = \left(\sum_{i=1}^n \overline{a}_{ki}e_i\right)\alpha_k = \alpha_k^*\alpha_k^*$. Thus $T_k^* C T_k = T_k^*(\alpha_k^*\alpha_k) = \alpha_k^*\hat{e}_k\alpha_k^*\alpha_k = \hat{e}_k(\alpha_k^*)\alpha_k^*\alpha_k = \overline{a}_{kk}\alpha_k^*\alpha_k = 2\alpha_k^*\alpha_k$ and therefore

$$R_k^* C R_k = (I - T_k)^* C (I - T_k) = C - T_k^* C - CT_k + T_k^* C T_k = C\,.\square$$

II. If we consider the non-diagonal entries of the matrix C as parameters we show that most groups $\Gamma = \langle R_1, R_2, \ldots, R_n \rangle$ are their free products. A typical element of Γ is a word $w = R_{i_1} R_{i_2} \ldots R_{i_m}$ where $R_{i_j} \neq R_{i_{j+1}}$, $i = 1, 2, \ldots, m-1$. We can consider now that w is a function on the parameter space of non-diagonal entries of C, namely.

$$w(C) = w(I - e_1\alpha_1, I - e_2\alpha_2, \ldots, I - e_n\alpha_n) = R_{i_1} R_{i_2} \ldots R_{i_m}$$

and is therefore given by polynomial functions w_{ij} of the entries of C. Thus the condition $w(C) = I$ can be expressed by the simultaneous equations $w_{ij} = 0$, $i \neq j$, $w_{ii} - 1 = 0$. These functions can not be identically zero on the parameters of C since there is a choice where the $\{R_i\}$ *do* generate their product and so we have the following.

Proposition 2. *There is a Zariski-dense set (of parameters a_{ij}) in $\mathbb{C}^{n^2-n}$ so that $R_i = I - \sum_{i=1}^{n} e_i\alpha_i$, $\alpha_i = \sum_{j=1}^{n} a_{ij} e_j$ generate their free product.*

Proof. It follows from Theorem 1 that each of the sets $w^{-1}(I)$ is an affine subvariety $\neq \mathbb{C}^{n^2-n}$; thus, $X = \mathbb{C}^{n^2-n} - \bigcup_w w^{-1}(I)$ is Zariski-dense. Thus the $\{R_i\}$ generate their free product on X. $\square$

III. Consider the set $V \times V^*$ for $V = \mathbb{C}^*$. Suppose that (v_1, α_1), (v_2, α_2), $\ldots$, (v_n, α_n) are elements of this set so that $\{v_1, \ldots, v_n\}$ is a basis for V and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V^*. We call this sequence a *framing* for $V \times V^*$ if $\alpha_i(v_i) = 2$, $i = 1, 2, \ldots, n$. Let $C = (\alpha_i(v_j))$, $T_i = v_i\alpha_i$ and $R_i = I - T_i$. Let Γ be the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (n-fold) and $G = \mathrm{Aut}(\Gamma)$. It is easy to see that G is generated by permutations and the automorphisms $\{g_{ij}\}$ where $g_{ij}(s_k) = \begin{cases} s_k & k \neq i \\ s_j s_k s_j & k = i \end{cases}$. Denote by PG, the kernel of $G \to \sum_n$, the pure automorphisms generated by $\{g_{ij}\}$ $1 \leq i \neq j \leq n$. If $g \in PG$, $g(s_k) = w_k s_k w_k$ for some elements $w_k \in \Gamma$. Define the action of PG on framings via

$$g((v_k, \alpha_k)) = (w_k(R_1, \ldots, R_n)v_k, \quad w_k(\hat{R}_1, \ldots, \hat{R}_n)\alpha_k)$$

where $\hat{R}_j\alpha_k = \alpha_k - \alpha_j(v_k)\alpha_j$. This gives a polynomial automorphism of the Cartan matrix.

For example if $g = g_{12}$, $g_{12}(s_1) = s_2 s_1 s_2$ and $g_{12}(s_j) = s_j$, $j \neq 1$ so $g_{12}(v_1, \alpha_1) = (R_2 v_1, \hat{R}_2\alpha_1)$ and $g_{12}(v_j, \alpha_j) = (v_j, \alpha_j)$, $j \neq 1$. Now $R_2 v_1 = v_1 - a_{21} v_2$ and $\hat{R}_2\alpha_1 = \alpha_1 - a_{12}\alpha_2$ and therefore the Cartan matrix for this framing is

$$\begin{pmatrix} 1 & -a_{12} & 0 & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -a_{12} & a_{13} - a_{12}a_{23} & \cdots \\ -a_{21} & 2 & a_{23} & \cdots \\ a_{31} - a_{21}a_{32} & a_{32} & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

§3 Three reflections

I. We now suppose that we are given three reflections corresponding to a Cartan matrix $C = \begin{pmatrix} 2 & a_{12} & a_{13} \\ a_{21} & 2 & a_{23} \\ a_{31} & a_{32} & 2 \end{pmatrix}$. The action of the group PG on C is given by:

$$g_{12}(C) = \begin{pmatrix} 2 & -a_{12} & a_{13} - a_{12}a_{23} \\ -a_{21} & 2 & a_{23} \\ a_{31} - a_{21}a_{32} & a_{32} & 2 \end{pmatrix}$$

$$g_{13}(C) = \begin{pmatrix} 2 & a_{12} - a_{13}a_{32} & -a_{13} \\ a_{21} - a_{31}a_{23} & 2 & a_{23} \\ -a_{31} & a_{32} & 2 \end{pmatrix}$$

$$g_{21}(C) = \begin{pmatrix} 2 & -a_{12} & a_{23} \\ -a_{21} & 2 & a_{23} - a_{21}a_{13} \\ a_{31} & a_{32} - a_{12}a_{31} & 2 \end{pmatrix}$$

$$g_{31}(C) = \begin{pmatrix} 2 & a_{12} & -a_{13} \\ a_{21} & 2 & a_{23} - a_{21}a_{13} \\ -a_{31} & a_{32} - a_{12}a_{31} & 2 \end{pmatrix}$$

$$g_{23}(C) = \begin{pmatrix} 2 & a_{12} - a_{13}a_{32} & a_{13} \\ a_{21} - a_{31}a_{23} & 2 & -a_{23} \\ a_{31} & -a_{32} & 2 \end{pmatrix}$$

$$g_{32}(C) = \begin{pmatrix} 2 & a_{12} & a_{13} - a_{12}a_{23} \\ a_{21} & 2 & -a_{23} \\ a_{31} - a_{23}a_{32} & -a_{32} & 2 \end{pmatrix}$$

Of course the trace and determinant of $g_{ij}(C)$ are the same as C, the second trace however is not invariant. The second trace $S_{ij}(C)$ of $g_{ij}(C)$

expressed in coordinates and the second trace of C, $S(C)$, is

$$S_{12}(C) = S(C) - a_{21}a_{32}a_{13} - a_{12}a_{23}a_{31} + a_{12}a_{21}a_{23}a_{32}$$
$$S_{13}(C) = S(C) - a_{13}a_{32}a_{21} - a_{12}a_{23}a_{31} + a_{13}a_{31}a_{23}a_{32}$$
$$S_{21}(C) = S(C) - a_{13}a_{32}a_{21} - a_{12}a_{23}a_{31} + a_{12}a_{21}a_{13}a_{31}$$
$$S_{31}(C) = S_{21}(C)\,, \quad S_{23}(C) = S_{13}(C)\,, \quad S_{32}(C) = S_{12}(C)\,.$$

Hence the second trace is invariant under PG in case:

$$a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31} = a_{12}a_{21}a_{13}a_{31} = a_{13}a_{31}a_{22}a_{32} = a_{12}a_{21}a_{23}a_{32}\,.$$

The variety defined by these three equations in the six variables is invariant under PG automorphisms. Restricting attention to this variety V we see that it is disjoint from the conditions imposed by Vinberg: $a_{ij} < 0$; so that no automorphism applied to a point in Vinberg's variety will be moved to a point of V.

We assume now that C is Hermitian. The variety $W : a_{23} = a_{12} = \overline{a}_{13}, a_{12}^3 + a_{13}^3 = a_{12}^2 a_{13}^2$ is invariant under all automorphisms of PG. Let $A = a_{12}$, $B = a_{13}$ (cf. [D-F-G]). According to our Theorem 1 we can show R_1, R_2, R_3 generate their free product if $|AB| \geq 4$, $|A^2| \geq 6|B|$, and $|B^2| \geq 6|A|$; but $\frac{A^2}{B} + \frac{B^2}{A} = AB$ so if $x = \frac{A^2}{B}$, $y = \frac{B^2}{A}$ then $x + y = xy$ or $y = \frac{x}{x-1}$ and this last conditions is impossible if $|x|, |y| \geq 6$; so the method of Theorem 1 fails miserably in this case. Since this variety W is PG invariant, it is again impossible to move a point of W to a point where the conditions of Theorem 1 apply.

Since the Cartan matrix C is assumed Hermitian we can diagonalize it and obtain representations of R_1, R_2, R_3 preserving this form. The characteristic polynomial of C is $-x^3 + 6x^2 - 3(4-AB)x + (4-AB)(2-AB) = y^3 - 3ABy + A^2B^2$ where $y = 2 - x$. This polynomial has three real roots since the product of the values at the critical points is negative. To see this it is useful to solve $A^3 + B^3 = A^2B^2$. Let $B = uA$ to obtain $A = u + u^{-2}$, $B = u^2 + u^{-1}$ where $|u| = 1$ and thus $AB = (u^{3/2} + u^{-3/2})^2$; put $u = e^{i\theta}$ and then $AB = 4\cos^2\frac{3\theta}{2} = 2(1 + \cos 3\theta)$ so that $4 - AB \geq 0$. The signature of the Hermitian form C is $(1,1,1)$ or $(1,1,-1)$ depending on whether $2 - AB > 0$ or $2 - AB < 0$ and $\det C \neq 0$.

Applying the Gram-Schmidt process to e_1, e_2, e_3 yields the three vectors $v_1 = e_1$, $v_2 = \left(-\frac{B}{2}, 1, 0\right)$, $v_3 = (\overline{\delta}, \delta, 1)$, $\delta = \frac{A^2-2B}{4-AB}$ so that $(v_1, v_1) = 2$, $(v_2, v_2) = \frac{4-AB}{2}$, $(v_3, v_3) = 2 - AB$ and $(v_i, v_j) = 0$, $i \neq j$. Thus if we put

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{4-AB}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2-AB}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{B}{2} & 1 & 0 \\ \overline{\delta} & \delta & 1 \end{pmatrix} \quad \text{we have that}$$

$C = (P^*P)^{-1}$ and thus $PR_i^*P^{-1}$ preserve the diagonalized form:

$$PR_1^*P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$PR_2^*P^{-1} = \begin{pmatrix} 1 - \frac{AB}{2} & -\frac{A}{2}\sqrt{4-AB} & 0 \\ -\frac{B}{2}\sqrt{4-AB} & -1 + \frac{AB}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$PR_3^*P^{-1} = \begin{pmatrix} 1 - \frac{AB}{2} & \frac{B\overline{\delta}}{2}\sqrt{4-AB} & -B\sqrt{\frac{2-AB}{2}} \\ \frac{A\delta}{2}\sqrt{4-AB} & 1 - \frac{AB}{2} & \delta\sqrt{\frac{(4-AB)(2-AB)}{2}} \\ -A\sqrt{\frac{2-AB}{2}} & \overline{\delta}\sqrt{\frac{(4-AB)(2-AB)}{2}} & AB - 1 \end{pmatrix}.$$

Finally one should observe that $S_1 = R_1R_2$, $S_2 = R_2R_3$ each have characteristic polynomial $(1-x)(x^2+(2-AB)x+1) = (1-x)(x^2 - 2\cos(3\theta)x+1) = (1-x)(x-\xi)(x-\overline{\xi})$ where $\xi = e^{i3\theta}$ and thus if ξ is an nth root of unity then $S_1^n = S_2^n = 1$ so that the representation is not faithful.

II. Degenerate Representations.

Consider the elements

$$X = R_1R_2 = \begin{pmatrix} AB-1 & A & A^2-B \\ -B & -1 & -A \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$Y = R_2R_3 = \begin{pmatrix} 1 & 0 & 0 \\ A^2-B & AB-1 & A \\ -A & -B & -1 \end{pmatrix}.$$

The common eigenspace of eigenvalue one for X and Y is the solutions to $CV = 0$. In the degenerate case $\det(C) = 0 = (4-AB)(2-AB)$ there are non-trivial solutions.

Proposition 3. *If* $\det C = 0$ *then* R_1, R_2, R_3 *generate an infinite abelian by finite group.*

Proof. Suppose $u = 1$ so that $A = B = 2$ then X and Y have common eigenvectors with eigenvalue 1, $v_1 = (1,-1,0)$ and $v_2 = (1,0,-1)$. Let $v_3 = (0,0,1)$ then we set a representation

$$X \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the image is free abelian of rank 2.

If $AB = 4$ and $u \neq 1$ then $u^3 = 1$ so that $u + u^{-1} = -1$, $A = 2u$, $B = 2u^{-1}$ and X, Y have common eigenspace $v_1 = (1,1,1)$, $v_2 = (1,0,-u)$ for eigenvalue 1. Let $v_3 = (0,1,0)$ then the representation is

$$X \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2u^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and the image is free abelian of rank 2.

If $AB = 2$ then $u^6 = 1$. The matrices X and Y have characteristic polynomial $(x-1)(x^2+1)$. We suppose that $u^3 = i$. Using the change of basis $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{B}{2} & 1 & 0 \\ \delta & \bar{\delta} & 1 \end{pmatrix}$ as in part I we obtain

$$R_1 \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \rightarrow \begin{pmatrix} 0 & -\frac{A}{2} & 0 \\ B & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_3 \rightarrow \begin{pmatrix} 0 & -\frac{B}{2}u^2 & 0 \\ -Au^{-2} & 0 & 0 \\ -A & -u^2 & 1 \end{pmatrix}$$

so that

$$X \rightarrow \begin{pmatrix} 0 & \frac{A}{2} & 0 \\ -B & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y \rightarrow \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ -A & -u^2 & 1 \end{pmatrix}$$

the vector $v_2 = (-\frac{A}{2}, i, 0)$ is a simultaneous eigenvector for eigenvalue $-i$ and with $v_1 = (0, 1-i, u^2)$ and $v_3 = (0,0,1)$ we obtain

$$X \rightarrow \begin{pmatrix} i & 0 & 0 \\ i-1 & -i & 0 \\ (1-i)u^2 & 0 & 1 \end{pmatrix}$$

$$Y \rightarrow \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$$

with image an extension of $\mathbb{Z}$ by $\mathbb{Z}_4$. □

References

[B-M] Bachmuth, S. and Mochizuki, H., *Triples of* 2×2 *matrices which generate free groups,* Proc. of Amer. Math. Soc. 59, 25–28 1976

[D-F-G] Dyer, J., Formanek, E. and Grossman, E., *On the linearity of automorphisms of free groups,* Archiv der Math. 38, 404–409, 1982

[H] Humphries, Stephen P., *Free subgroups of* $SL_n(\mathbb{Z}), n > 2$ *generated by transvections,* J. of Algebra 116, 115-162 (1988)

[V] Vinberg, E. B., *Discrete linear groups generated by reflections,* Math. USSR Izvestija, 5 no. 5, 1083–1119 (1991)