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Zariski Surfaces
Part I
Definition and General Properties. The Theory of Adjoints

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§0. Introduction

In the classical case all unirational surfaces are rational. This was first realized by Oscar Zariski ([20], p. 314). Prompted by Hironaka's suggestion, we began an investigation of the type of surfaces introduced by Zariski in that paper. The research was originally begun by Blass in 1970-71 at Harvard with the advice of Hironaka and Zariski, and then during 1974–1977 he continued under the direction of J. S. Milne and M. Hochster at the University of Michigan. The two remaining authors entered this study a bit later.

A smooth algebraic surface X defined over an algebraically closed field, k, of characteristic $p > 0$ is called a Zariski Surface (or ZS) if and only if there exists two elements x, y in the function field of X, denoted $k(X)$, that are algebraically independent over k and such that

$$k(x, y) \subseteq k(X) \subseteq k(x^{1/p}, y^{1/p}).$$

The main results of the paper are as follows. First of all, section 3 answers a question posed by Zariski in 1970-71. He asked whether a ZS

with vanishing geometric genus, p_g, is necessarily rational. A long counterexample is given in section 3. Secondly, in section 5 it is shown that the value of p_g is unbounded over an algebraically closed field of characteristic larger than or equal to 5. This, together with some other results in section 3, illustrates the richness of the class of ZS.

In section 4 a more detailed study is made of a particularly simple subclass of ZS's which we call "generic" ZS's. (A generic ZS is a smooth minimal model of the function field of a hypersurface given by $z^p = f(x, y)$ where f has degree $p + 1$ and the hypersurface has only the simplest singularities.) We determine p_a, p_g, $\mathbb{P}_2$, $\mathbb{K}^2$ for a generic Zariski surface as well as the rank of the Néron-Severi group ρ and the e'tale Betti numbers, b_i. Using "generic Zariski surfaces", we give examples of ZS's which are of general type and $K3$. Trivially there also exist rational ZS's.

All Zariski surfaces are unirational and consequently supersingular, i.e., $\rho = b_2$. Thus the richness of the class of ZS's in characteristic $p > 0$ is in sharp contrast to the situation in characteristic 0 where every unirational surface is well known to be rational.

The principal technical tool used in the paper is the theory of adjoints and multi-adjoints. This theory deals with the influence of singularities on differential forms and is classical. However, since no reference could be found for the results that we needed, a self-contained exposition of the facts from the theory that we use is given in section 2. We develop the theory of adjoints for a normal, two-dimensional hypersurface in affine or projective 3-space. The results about adjoints are proved over an arbitrary algebraically closed field of arbitrary characteristic.

ZS's are an interesting subclass of unirational surfaces. Among the many open problems concerning ZS's, we mention two very concrete ones. The first one is to answer Zariski's questions in characteristic $p > 0$. The second one is whether $H^1(X, O_X)$ can be nontrivial for a Zariski surface X. Added in revision: W. E. Lang gave an example of a ZS with H^1 nontrivial. He also settled by another example Zariski's question in characteristic 3 [8].

Notation.

- k – algebraically closed field of characteristic $p > 0$, unless stated to the contrary.
- A^n – affine n-space over k.
- P^n – projective n-space over k.

Surface – irreducible, reduced, quasi-projective variety over k.
If X denotes a surface then $k(x)$ denotes its function field.
For a smooth surface X we will write as usual:

- p_g = geometric genus of $X = \dim H^2(X, O_X)$.
- p_a = arithmetic genus of $X = p_g - \dim H^1(X, O_X)$.
- $p_i = \dim H^0(X, K^{\otimes i})$, where K is the canonical line bundle on X.
- $p_1 = p_g$ by Serre duality.
- q = dim of the Picard variety of X = dim of the Albanese variety of X.

The notation:

$F\colon f(x,y,z)=0$ means $F=\operatorname{Spec}\frac{k[x,y,z]}{(f(x,y,z))}$; $F\subseteq A^3$. If $F(x,y,z,z_0)$ is a homogeneous form:

$\bar{F}\colon F(x,y,z,z_0)=0$ means $\bar{F}=\operatorname{Proj}\frac{k[x,y,z,z_0]}{(F(x,y,z,z_0))}$; $\bar{F}\subseteq P^3$.

If X is a surface we denote by $\mathbf{Sing(X)}$ the subscheme consisting of singular points of X. X' denotes $X-\operatorname{Sing}(X)$. A *desingularization of* $\mathbf{X}$ is a proper, birational surjective morphism $\pi\colon\widetilde{X}\to X$ satisfying the following two conditions:

i) $\widetilde{X}$ is smooth, and

ii) $\pi^{-1}(X')\xrightarrow{\pi|\pi^{-1}(X')}X'$ is an isomorphism.

If X is a surface and $p\in X$ is a closed point, then we refer to *blowing up the point* p as "quadratic transformation with center p".

Let I be an ideal in a ring R. $r\in R$ is said to be *integrally independent on* I if there exists an equation

$$r^n+a_1r^{n-1}+a_2r^{n-2}+\cdots+a_n=0 \quad \text{with} \quad a_i\in I^i.$$

I is *integrally closed* if it contains all elements of R integrally dependent on I (see [13], p. 34). The reader is referred to ([2], [9]) for the definition and basic properties of rational singularities.

§1. Zariski Surfaces: definition and general properties

Zariski surfaces are defined and some of their simplest properties are given in this section. X denotes a smooth projective surface over k, $k(X)$ is the function field of X, and $k=\bar{k}$, $\operatorname{char}(k)=p>0$.

Definition 1.1.0. A smooth projective surface X is called a *Zariski surface* if there exists two elements x, y in $k(X)$ that are algebraically independent over k such that $k(x,y)\subseteq k(X)\subseteq k(x^{1/p},y^{1/p})$.

Remark 1.1.1. Unless one of the inclusions in (1.1.0) is an equality, the extensions $k(X)$ over $k(x,y)$ and $k(x^{1/p},y^{1/p})$ over $k(X)$ are both purely inseparable of degree p.

Remark 1.1.2. We propose to call a surface a *singular Zariski surface* if it satisfies (1.1.0) but is not necessarily smooth.

Remark 1.1.3. Definition (1.1.0) is birational, i.e., it depends only on the function field; thus any smooth projective surface birationally equivalent to a Zariski surface is itself a Zariski surface. Two surfaces are birationally equivalent if they have isomorphic function fields over k.

Remark 1.1.4. A Zariski surface is unirational by definition.

Notation 1.1.5. We shall sometimes abbreviate "Zariski surface" by ZS.

Proposition 1.2.0. *X is a Zariski surface if and only if X is birationally equivalent to a hypersurface in A^3 defined by an irreducible equation of the form $z^p - f(x,y) = 0$ where $f(x,y) \in k[x,y]$, the polynomial ring in two variables over k.*

Proof. Suppose that X is birationally equivalent to $\operatorname{Spec}(k[x,y,z]/(z^p - f(x,y))$ with $z^p - f(x,y)$ irreducible. Then $k(X) \cong k(\bar{x},\bar{y},\bar{z})$ where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the residue classes in $k[x,y,z]/(z^p - f(x,y))$ of x, y, z, respectively. We have, by standard field theory,

$$k(x,y) \cong k(\bar{x},\bar{y}) \subsetneq k(\bar{x},\bar{y},\bar{z}).$$

Since $\bar{z} = f_1(\bar{x}^{1/p}, \bar{y}^{1/p})$, where f_1 is obtained from f by taking p-th roots of all the coefficients, we have $k(\bar{x},\bar{y},\bar{z}) \subseteq k(\bar{x}^{1/p}, \bar{y}^{1/p})$. Since $[k(\bar{x},\bar{y})\colon k(\bar{x}^{1/p},\bar{y}^{1/p})] = p^2$ and $\bar{z}^p \in k(\bar{x},\bar{y})$, we see that $k(\bar{x},\bar{y}) \nsubseteq k(\bar{x},\bar{y},\bar{z}) \nsubseteq k(\bar{x}^{1/p},\bar{y}^{1/p})$, so that X is a ZS.

Suppose next that X is a Zariski surface. Then there exists $x, y \in k(X)$ such that $k(x,y) \subseteq k(X) \subseteq k(x^{1/p}, y^{1/p})$. If either containment is not proper, then $k(X)$ is birationally equivalent to the hypersurface $F\colon z^p - x$. Thus, we may assume that both containments are proper. Let $z \in k(X) - k(x,y)$. Then $z^p = w(x,y)/v(x,y)$ with $w, v \in k[x,y]$ and w/v not a p-th power of an element in $k(x,y)$. $z^p = w/v$ implies $z_1^p = wv^{p-1}$ where $z_1 = z \cdot v \in k(X) - k(x,y)$. Since w/v is not a p/-th power in $k(x,y)$, niether is wv^{p-1} a p-th power in $k[x,y]$. Since $k(x,y) \subsetneq k(x,y,z_1) \subseteq k(X)$ and $[k(X)\colon k(x,y)] = p$, we obtain $k(X) = k(x,y,z_1)$. It follows that X is birationally equivalent to

$$\operatorname{Spec} \frac{k[\bar{x},\bar{y},\bar{z}_1]}{(\bar{z}_1^p - w(\bar{x},\bar{y})v(\bar{x},\bar{y})^{p-1})}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}_1$ are indeterminants and the equation is irreducible. □

Proposition 1.3.0. *Given any irreducible, reduced hypersurface in A^3, $F\colon z^p = f(x,y)$, there exists a Zariski surface $\widetilde{F}$ birationally equivalent to F.*

Proof. Let $\bar{F}$ be the closure of F in $\mathbb{P}^3$. Let $\widetilde{F}$ be a desingularization of $\bar{F}$, which Abhyankar proved exists [1]. Then $\widetilde{F}$ is a smooth projective surface which is birationally equivalent to F so it is a ZS by (1.2.0).

Remark 1.3.1. F, $\bar{F}$, $\widetilde{F}$ will often be given the above meaning throughout the paper.

Remark 1.3.2. $\widetilde{F}$ is not unique. However, if we require $\widetilde{F}$ to be a relatively minimal model of $k(F)$ then $\widetilde{F}$ will be unique unless $\widetilde{F}$ is ruled or rational ([18], [19]). It will be proved later that $q = \dim \operatorname{Alb}(\widetilde{F}) = 0$ (see (1.6.0)), where $\operatorname{Alb}(\widetilde{F})$ denotes the Albanese variety of $\widetilde{F}$. It follows that $\widetilde{F}$ ruled implies $\widetilde{F}$ rational.

Proposition 1.4.0. *Let t_1, t_2 be indeterminates. Let L be a field with $k(t_1, t_2) \subseteq L \subseteq k(t_1^{1/p}, t_2^{1/p})$. Then there exists a ZS X with $k(X) \approx L$.*

Proof. Replace $k(X)$ by L in the proof of the "only if" direction of Proposition (1.2.0) to obtain a hypersurface $F\colon z^p = f(x, y)$ with $k(F) \cong L$. Now use (1.3.0). □

Proposition 1.5.0. *Let k be as above, t_1, t_2 indeterminates over k. Let $\mathcal{L}$ be the set of all fields L such that $k(t_1, t_2) \subseteq L \subseteq k(t_1^{1/p}, t_2^{1/p})$. Let $\mathcal{PS}$ be the set of all (projective smooth) Zariski surfaces over k. We have a one-to-one correspondence*

$$\mathcal{L}/(\text{field isomorphisms over } k) \leftrightarrow \mathcal{PS}/(\text{birational equivalence}).$$

Proof. By (1.4.0) and the definitions. □

Remark 1.5.1. We shall use (1.5.0) to show that the set $\mathcal{L}/$ (field isomorphisms over k) is infinite when $p \geq 5$ (see (5.1.2).)

Lemma 1.6.0. *The dimension of the Albanese variety of a Zariski surface is 0. In fact, the Albanese variety is trivial.*

Proof. Since Zariski surfaces are unirational, the proof follows from B. Lang ([8], corollary, p. 25), where a unirational variety is semi-pure. □

Lemma 1.7.0. The Picard variety of a Zariski surface is trivial.

Proof. Let $\widetilde{F}$ be a Zariski surface. By (1.6.0) the Albanese variety of $\widetilde{F}$, $\mathrm{Alb}(\widetilde{F})$, is trivial. The Picard variety of $\widetilde{F}$ is the same as that of $\mathrm{Alb}(\widetilde{F})$ (Lang, [8], p. 148, Theorem 1), but the Picard variety of the trivial variety is trivial. □

Proposition 1.8.0. *If a non-singular Zariski surface $\widetilde{F}$ satisfies $P_2(\widetilde{F}) = 0$, then $\widetilde{F}$ is rational.*

Proof. This proof is adapted from [20], p. 314 and is given here for the reader's convenience. By (1.6.0) and (1.7.0) dim $\mathrm{Alb}(\widetilde{F})$ = dim Picard variety of $\widetilde{F} = 0$. Also $P_2(\widetilde{F}) = 0$ implies $p_g = 0$. Nakai has shown that if $p_g = 0$ then the dimension of the Picard variety is equal to the dimension of $H^1(\widetilde{F}, \mathcal{O}_{\widetilde{F}})$ (see [14], Theorem 5). Thus $p_g - p_a = \dim H^1(\widetilde{F}, \mathcal{O}_{\widetilde{F}}) = 0$ and hence $p_a = p_g = 0$. Also $P_2 = 0$ by assumption and the rationality of $\widetilde{F}$ follows from Castelnuovo's criterion of rationality (see [20]). □

Proposition 1.9.0. *Let X be a ZS. Let b_i = rank of $H^i_{et}(X, Q_\ell)$ where $\ell \neq p$. Then $b_0 = b_4 = 1$ and $b_1 = b_3 = 0$.*

Proof. Since $q = \dim \mathrm{Alb}(X) = 0$, the proposition follows from [10], Ch. V and Ch. VI.11. □

Remark 1.10.0. Using the method of proof of p. 286 of [15], one can show that the e'tale fundamental group of a Zariski surface is trivial.

§2. The Theory of Adjoints

The theory of adjoints is well known and classical for algebraic curves. For surfaces it was developed by Clebsch, M. Noether, the Italian geometers and Zariski. However, it is difficult to find a complete rigorous treatment of the theory for a two-dimensional, normal, hypersurface. Without a doubt that theory has been well known to several writers in the theory of algebraic surfaces, but no reference could be found for the proofs of the results they used. It is the purpose of this section to fill this gap. The treatment here relies only on rudiments of sheaf theory. For a more homological treatment of adjoints see [4]. We also discuss briefly the notion of multiadjoints and connections with plurigenera – a favorite topic with Enrique's but again a modern reference is missing. The theory of adjoints and multiadjoints is our principal tool in all later sections.

Throughout this section, k will be an algebraically closed field of any characteristic. All surfaces will be assumed to be quasiprojective, irreducible and reduced. Surfaces will be denoted by capital letters V, W, X, Y, etc. $k(X)$ denotes the function field of a surface X. $\Omega^1_{k(x)/k}$, $\Omega^2_{k(x)/k}$ are the Kähler differentials of $k(X)$ over k (respectively, two-forms). The letters p, q always denote *closed* points.

Proposition 2.1.0. *$\Omega^2_{k(x)/k}$ is a one-dimensional vector space over $k(X)$. If $a, b \in k(X)$ are a separating transcendence basis, $da\, db \neq 0$ in $\Omega^2_{k(X)/k}$ and $da\, db$ is a basis over $k(X)$ ($d\colon k(X) \to \Omega^2_{k(X)/k}$ denotes the usual differential map).*

Proof. Well known.

Remark 2.1.1. $a \in k(X)$, $\alpha \in \Omega^2_{k(X)/k}$, $a \neq 0$, $\alpha \neq 0$ implies a $\alpha \neq 0$ in $\Omega^2_{k(X)/k}$.

Let X be a surface and $X' = X - \operatorname{Sing} X$. If $p \in X'$ is a closed point, we define a subset of $\Omega^2_{k(X)/k}$ which we shall call $\operatorname{Reg}_X(p)$ as follows:

Definition 2.1.2. $\alpha \in \operatorname{Reg}_X(p)$ if and only if $\alpha = \Sigma a_i db_i dc_i$, with $a_i, b_i, c_i \in (\mathcal{O}_x)_p$, where $(\mathcal{O}_x)_p$ is the local ring of p.

Proposition 2.2.0. *If $p \in X'$ then $Reg_X(p)$ is an $(\mathcal{O}_x)_p$-module. It is free of rank one. If t_p, t'_p are a regular system of parameters at p then $dt_t dt'_p$ is a generator of $Reg_X(p)$ as an $(\mathcal{O}_x)_p$-module.*

Proof. Well known.

Using the above notions we proceed to define a sheaf ω_X on X which is isomorphic to the sheaf of Kähler two-forms if X is smooth. In the general case considered here $\omega_X = j_*$ (sheaf of Kähler two-forms on X') where $j: X' \to X$ is an open immersion.

First we define a presheaf of subsets of $\Omega^2_{k(x)/k}$. Let $U \subseteq X$ be open; define

$$\omega_X(U) = \bigcap_{p \in U'} \mathrm{Reg}_X(p).$$

If $V \subset U$, $\omega_X(V) \supset \omega_X(U)$ and we define the restriction map $\alpha^u_v: \omega_X(U) \to \omega_X(V)$ to be the inclusion map.

Proposition 2.3.0. *ω_X is a sheaf of $\mathcal{O}_x$-modules. All restriction maps are monomorphisms and k-linear.*

Proof. ω_X is clearly a presheaf of $\mathcal{O}_x$-modules. To see that ω_X is a sheaf, assume $V = \bigcup_{i \in I} V_i$ is an open cover of an open subset V of X and $\alpha_i \in \omega_X(V_i)$ such that $\alpha_i|_{v_i \cap v_j} = \alpha_j|_{v_i \cap v_j}$ for each $i, j \in I$. Then this implies that $\alpha_i = \alpha_j$ in $\omega_X(V_i \cap V_j)$, for each $i, j \in I$. For a fixed i this yields $\alpha_i \in \omega_X(V_j)$ and $\alpha_i = \alpha_j$ for all $j \in I$. Thus $\alpha_i \in \bigcap_{j \in I} \omega_X(V_j) = \omega_X(U)$ and $\alpha_i|_{v_j} = \alpha_j$. Therefore ω_X is a sheaf of $\mathcal{O}_x$-modules as well. $\square$

Definition 2.4.0. Let $X \subset X'$. Define $\omega_X(S) = \bigcap_{q \in S} \mathrm{Reg}_X(q)$.

Note 2.5.0. If $a \in \bigcap_{q \in S} (\mathcal{O}_X)_q$, then $a\omega_X(S) \subseteq \omega_X(S)$.

Behavior of ω_X Under Restriction to Open Subschemes

Proposition 2.6.0. *Suppose U is a nonempty open subset of a surface X. We identify $k(U) = k(X)$ and $\Omega^2_{k(U)/k} = \Omega^2_{k(X)/k}$. Then $\omega_u(V) = \omega_X(V)$ for each $V \subseteq U$.*

Proof. The first equality is trivial; the second follows from (2.6.0). $\square$

Behavior of ω_X Under Certain Birational Maps

Discussion 2.7.0. Let W, Z be surfaces. Let $W \xrightarrow{\pi} Z$ be a birational map satisfying the following condition

$$\pi^{-1}(Z') \xrightarrow{\pi|\pi^{-1}(Z')} Z' \qquad \text{is an isomorphism.}$$

Then we have an isomorphism $k(Z) \xrightarrow{\pi_1} k(W)$ and π_1 induces an isomorphism $\Omega^2_{k(Z)/k} \xrightarrow{\pi_1} \Omega^2_{k(W)/k}$ of k-vector spaces. We denote the inverse isomorphism by π_2, so that $\Omega^2_{k(W)/k} \xrightarrow{\pi_2} \Omega^2_{k(Z)/k}$.

Proposition 2.8.0. *Let $W \xrightarrow{\pi} Z$ be as in (2.7.0). Then $\pi_2(\omega_W(\pi^{-1}(U))) \subseteq \omega_z(U)$ for every open $U \subseteq Z$.*

Proof. Let $U \subset Z$ be open. Since $\pi^{-1}(Z') \to Z'$ is an isomorphism π_2 maps $\omega_W(\pi^{-1}(U'))$ isomorphically onto $\omega_z(U)$ by (2.6.0). The result now follows from the fact that $\pi^{-1}(U) \supseteq \pi^{-1}(U')$ and the restriction maps on ω_W are inclusions. $\square$

Remark 2.8.1. $\pi_2\colon \omega_W(\pi^{-1}(U)) \to \omega_z(U)$ is a monomorphism of k-vector spaces. In fact, π_2 defines a monomorphism of $\mathcal{O}_z$-modules $\pi_*\omega_W \rightsquigarrow \omega_z$.

Affine Case, Adjoints

Assumption 2.8.2. Let us assume that $F \subseteq \mathbb{A}^3$ is a normal, irreducible, reduced hypersurface. Throughout this subsection F will have this meaning. Let us describe ω_F.

Proposition 2.9.0. *There exists a differential $\sigma_F \in \omega_F(F)$ such that for every $U \subseteq F$ open, $\omega_F(U) = \mathcal{O}_F(U)\sigma_F \subseteq \Omega^2_{k(F)/k}$ is a free $\mathcal{O}_F(U)$-module on one generator, σ_F, and such that for every point $p \in F'$, $\sigma_F \in Reg_F(p)$ and $Reg_F(p)$ is a free $(\mathcal{O}_F)_p$-module on one generator, namely σ_F.*

The differential σ_F is unique up to multiplication by a unit in $\mathcal{O}_F(F)$. σ_F will be called a canonical generating differential (abbreviated c.g.d.) on F. Moreover, if we write $F = Spec\, k[x_1, x_2, x_3]/(f(x_1, x_2, x_3))$ and $\frac{\partial f}{\partial x_i} \not\equiv 0$ then we may take $\sigma_F = \frac{dx_j dx_k}{\partial f/\partial x_i}$, i, j, k distinct.

Proof. The proposition is well known, see for instance [16], Ch. III, §5.4, where non-singularity is assumed but is not essential to the proof.

Remark 2.10.0. $\sigma_F \neq 0$.

Lemma 2.11.0. $a \notin (\mathcal{O}_F)_p$ implies $a\sigma_F \notin \mathrm{Reg}_F(p)$.

Proof. Suppose $a\sigma_F \in \mathrm{Reg}_F(p)$. Then $a\sigma_F = a'\sigma_F$, $a' \in \mathcal{O}_F(p)$. Then $(a - a')\sigma_F = 0$, hence $a - a' = 0$, which shows $a \in (\mathcal{O}_F)_p$. $\square$

Definition 2.12.0. Let F be as in (2.8.2). Let $p \in F$ be an isolated singular point. Let $\widetilde{F} \xrightarrow{\pi} F$ be any map such that

a) π is birational and proper.

b) $\pi|\pi^{-1}(F')$ is an isomorphism.

c) $\pi^{-1}(p) \subseteq \widetilde{F}'$.

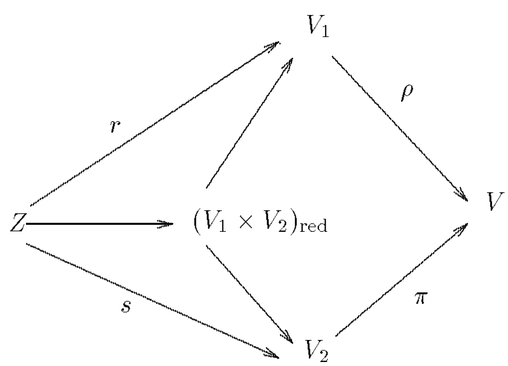
We call π a *resolution of p in F*.

Definition 2.13.0. Let F, p be as in (2.12.0). Let σ_F be a canonical generating differential on F. We call $a \in (\mathcal{O}_F)_p$ an adjoint (locally) at p if there exists a resolution $\widetilde{F} \xrightarrow{\pi} F$ of p in F such that $\pi_1(a\sigma_F)$ is regular on $\pi_1^{-1}(p)$ (or, equivalently, $\pi_1(a\sigma_F) \in \mathrm{Reg}_{\widetilde{F}}(\pi^{-1}(p))$).

Remark 2.13.1. Let σ'_F be another generating differential on F. We have $\sigma'_F = u\sigma_F$, $u \in \mathcal{O}_F(F)$ a unit; then $\pi_1(a\sigma'_F) = \pi_1(u)\pi_1(a\sigma_F)$. Since $\pi_1(u)$ is regular on $\pi^{-1}(p)$, $\pi_1(a\sigma'_F)$ is regular on $\pi^{-1}(p)$. This shows (2.13.0) is independent of the choice of σ_F.

Proposition 2.14.0 (Independence of the resolution)**.** *Again, let F, p be as in (2.12.0) and let $a \in \mathcal{O}_p$ be an adjoint at p. Let $G \xrightarrow{\rho} F$ be any resolution of p in F; then for any choice of a canonical generating differential σ_F, we have $\rho_1(a\sigma_F) \in Reg(\rho^{-1}(p))$.*

Proof. Consider an open neighborhood V of p such that $V = V' \cup \{p\}$. Let $\widetilde{F} \xrightarrow{\sim} F$ be a resolution as in (2.13.0), which arises from $a \in (\mathcal{O}_F)_p$ being an adjoint. Let $V_1 = \rho^{-1}(V)$, $V_2 = \pi^{-1}(V)$; V_1, V_2 are smooth. We have the following diagram where Z is a desingularization of $(V_1 \times V_2)_{\text{red}}$. It is easy to see that r and s are both birational and proper. If $\rho_1(a\sigma_F) \notin \text{Reg}(\rho^{-1}(p))$, then $\rho_1(a\sigma_F)$ has a polar curve on V_1. By ([15], page 55), r is a composition of finitely many quadratic transforms with point centers. Thus $r_1\rho_1(a\sigma_F)$ cannot be regular on Z.



However, since $\pi_1(a\sigma_F)$ is regular on V_2 and s is everywhere defined, we see that $s_1\pi_1(a\sigma_F)$ is regular on Z. But $r_1\rho_1(a\sigma_F) = s_1\pi_1(a\sigma_F)$; a contradiction. □

Remark 2.14.1. The above argument is adapted from the one used by Grauert-Reimenschneider in the analytic case ([6], p. 271).

Remark 2.14.2. If G is isomorphic to an affine normal surface $F \subseteq A^3$, we can define in the obvious way what is meant by an adjoint at an isolated singularity of G. One easily sees that this notion is independent of the choice of F and of the isomorphism.

Theorem 2.15.0. *Let G be isomorphic to $F \subseteq A^3$, F as in (2.8.3). Let $p \in G$ be an isolated singularity. Then the adjoints in $(\mathcal{O}_G)_p$ form an ideal of finite co-length. Moreover, if $G = F\colon f(x_1, x_2, x_3) = 0$, then $\partial f/\partial x_i$ is adjoint for every $i = 1, 2, 3$.*

Proof. We may assume $G = F$ and $F = \text{Spec}\,\frac{k[x_1,x_2,x_3]}{(f(x_1,x_2,x_3))}$. Let $\pi\colon \widetilde{F} \to F$ be a resolution of p in F. The partial $\partial f/\partial x_i$ is adjoint if $\partial f/\partial x_i \equiv 0$. If $\partial f/\partial x_i \not\equiv 0$, then $\sigma_F^{(i)} = \frac{dx_j\,dx_k}{\partial f/\partial x_i}$ is a canonical generating differential on F, for $0 \leqq i, j, k \leqq 3$ all distinct. Then we have $(\partial f/\partial x_i) \cdot \sigma_F^{(i)} = dx_j dx_k$. Since $x_j, x_k \in (\mathcal{O}_F)_p$,

$$\pi_1(x_j), \pi_1(x_k) \in \bigcap_{q \in \pi^{-1}(p)} (\mathcal{O}_{\widetilde{F}})_q.$$

Thus

$$\pi_1((\partial f/\partial x_1)\sigma_F^{(i)}) = d\pi_1(x_j) d\pi_1(x_k) \in \mathrm{Reg}_F(\pi^{-1}(p)),$$

which shows that $\partial f/\partial x_i$ is adjoint.

Let us show that the adjoints in $(\mathcal{O}_F)_p$ form an ideal. The proof is standard. Suppose $a, b \in (\mathcal{O}_F)_p$ and b is adjoint. For any canonical generating differential σ_F, $\pi_1(ab\sigma_F) = \pi_1(a)\pi_1(b\sigma_F)$. Since $\pi_1(a)$ is regular on $\pi^{-1}(p)$ and $\pi_1(b\sigma_F) \in \mathrm{Reg}_F(\pi^{-1}(p))$ by (2.14.0), $\pi_1(ab\sigma_F) \in \mathrm{Reg}_{\widetilde{F}}(\pi^{-1}(b))$ and ab is adjoint.

If a, b are adjoint in $(\mathcal{O}_F)_p$, then $a+b$ is trivially. Thus the adjoints form an ideal in $(\mathcal{O}_F)_p$ which we will call $\mathrm{Adj}(p)$. Since $p \in F$ is an isolated singularity, the ideal $J_p = (\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3)$ has finite co-length. Since $J_p \subseteq \mathrm{Adj}(p)$, the latter has finite co-length. □

Terminology 2.15.1. We call $e_p = \dim_k(\mathcal{O}_F)_p/\mathrm{Adj}(p))$ the *Noether-Enriques number of the singularity.* It is often denoted p and sometimes called the "genus". For surfaces this invariant seems first to have been studied by Max Neother and then taken up by the Italian school; Enriques discussed a number of special cases in his book "le Superficie Algebriche" [5].

Proposition 2.16.0. *Let $G \subset X$ be an open subscheme isomorphic to $F \subseteq A^3$ where F is as in (2.8.3). Let σ_G be a canonical generating differential. Let $\widetilde{X}$ be a desingularization of $X, \widetilde{X} \xrightarrow{\pi} X, \pi^{-1}(G) = \widetilde{G}$. Let Sing$(G) = \{p_1, \ldots, p_s\}$. Then $\pi_2(\omega_{\widetilde{X}}(\widetilde{G})) = (\mathcal{O}_X(G) \cap \bigcap_{i=1}^{s} \mathrm{Adj}(p_i))\sigma_G$.*

Proof. We can easily reduce to the case where $G = X$, $G \subseteq A^3$ is affine and $\widetilde{G} \xrightarrow{\pi} G$ is a desingularization. We need to show

$$\pi_2(\omega_{\widetilde{G}}(\widetilde{G})) = \left(\mathcal{O}_G(G) \cap \bigcap_{i=1}^{s} \mathrm{Adj}(p_i)\right)\sigma_G.$$

Let α belong to the right hand side. Then $\alpha = a\sigma_G$, $a \in \mathcal{O}_G(G)$,w here a is adjoint at all the singular points of G. Then $\pi_1(\propto) \in \omega_{\widetilde{G}}(\widetilde{G})$, so $\propto = \pi_2(\pi_1 \propto) \in \pi_2(\omega_{\widetilde{G}}(\widetilde{G}))$. Suppose $\beta \in \pi_2(\omega_{\widetilde{G}}(\widetilde{G}))$. Then $\pi_1(\beta) \in \omega_{\widetilde{G}}(\widetilde{G})$. Therefore β is regular on G', so $\beta \in \omega_G(G') = \mathcal{O}_G(G')\sigma_G$. But since G is normal and $G - G'$ is a finite set of points, we have $\mathcal{O}_G(G') = \mathcal{O}_G(G)$. Hence $\beta = a\sigma_G$ with $a \in \mathcal{O}_G(G)$. Moreover, since $\pi_1(\beta)$ is regular on $\pi^{-1}(p_j)$, $j = 1, 2, \ldots, s$, a is adjoint at all the singular points of G, so that $a \in \mathcal{O}_G(G) \cap \bigcap_{i=1}^{s} \mathrm{Adj}(p_j)$. Thus β belongs to the right hand side above. □

For future reference, in section 3 we introduce the definition of an adjoint surface.

Definition 2.16.1. Let $F = \text{Spec}\,\frac{k[x_1,x_2,x_3]}{(f(x_1,x_2,x_3))}$ be as in (2.8.3). let $p \in F$ be an isolated singularity. Let $H = \text{Spec}\,\frac{k[x_1,x_2,x_3]}{(A(x_1,x_2,x_3))}$ be another closed subscheme of A^3. We say H is an *adjoint surface* to F at p if $A(x_1,x_2,x_3)$ considered as an element in $k[F]$ is an adjoint of the local ring of F at p.

Remark 2.16.2. An "adjoint surface" is not assumed to be irreducible or reduced. It need not be a surface in the sense of this paper.

Projective Case

Definition 2.17.0. Assume $X = \text{Proj}\,\frac{k[X_0,X_1,X_2,X_3]}{(F(X_0,X_1,X_2,X_3))}$ where the degree of F is $n \geq 4$; X is assumed to be reduced, irreducible, and normal. Let us denote by $k[X] = k[x_0,x_1,x_2,x_3]$ the graded ring $\frac{k[X_0,X_1,X_2,X_3]}{(F(X_0,X_1,X_2,X_3))}$ and by $k[X]_{(d)}$, $d = 0,1,2,\ldots$, the d-th graded piece of $k[X]$.

As usual, we identify $k(X)$ with the set of ratios a/b where $a, b \in k[X]_d$ for some d. X is covered by four affines

$$X_i = \text{Spec}(k[X]_{x_i})_0 = \text{Spec}\, k\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \frac{x_2}{x_i}, \frac{x_3}{x_i}\right].$$

It is important to note that each of the X_i's is isomorphic to an affine surface. Explicitly, we have, for example,

$$X_0 \approx H_0 = \text{Spec}\,\frac{k[Y_1,Y_2,Y_3]}{(F(1,Y_1,Y_2,Y_3))} = \text{Spec}\, k[y_1,y_2,y_3],$$

the isomorphism of kH_0 with $k[x_1/x_0, x_2/x_0, x_3/x_0]$ being given by $y_1 \to x_1/x_0$; $y_2 \to x_2/x_0$; $y_3 \to x_3/x_0$; and similarly for $i = 2,3,4$.

Since X is reduced and normal, at least two of the partial derivatives $\partial F/\partial X_i$ are not identically 0. After renumbering the variables, we may assume the following condition.

2.17.1. $\partial F/\partial X_0 \not\equiv 0$ and $\partial F/\partial X_1 \not\equiv 0$ in the polynomial ring $k[X_0, X_1, X_2, X_3]$ where the X_i are indeterminates. Then it is easy to show that $\partial F/\partial X_0$ and $\partial F/\partial X_1 \in k[X]_{(n-1)}$ are also not identically 0 in $k[X]$.

We fix the following canonical generating differentials on X_0, X_1, X_2, and X_3,

$$\sigma_0 = \frac{d(x_2/x_0)d(x_3/x_0)}{\partial F/\partial X_1(1, x_1/x_0, x_2/x_0, x_3/x_0)} = \frac{d(x_2/x_0)d(x_3/x_0)}{(\partial F/\partial X_1)/x_0^{n-1}},$$

$$\sigma_1 = \frac{d(x_3/x_1)d(x_2/x_1)}{(\partial F/\partial X_0)/x_1^{n-1}},$$

$$\sigma_2 = \frac{d(x_3/x_2)d(x_0/x_2)}{(\partial F/\partial X_1)/x_2^{n-1}} = \frac{d(x_1/x_2)d(x_3/x_2)}{(\partial F/\partial X_0)/x_2^{n-1}}$$

(This second equality is easily justified. See [16], ch. VII 5.4)

$$\sigma_3 = \frac{d(x_0/x_3)d(x_2/x_3)}{(\partial F/\partial X_1)/x_3^{n-1}}.$$

Key Computation Lemma 2.18.0. $\sigma_i = (x_i/x_j)^{n-4}\sigma_j,\ 0 \le i, j \le 3,$ *in* $\Omega^2_{k(X)/k}$.

Proof. First compare σ_0 and σ_3.

$$\begin{aligned}
\frac{\sigma_0}{(x_0/x_3)^{n-1}} &= \frac{d(x_2/x_0)d(x_3/x_0)}{\frac{\partial F}{\partial X_1}(x_0, x_1, x_2, x_3) \cdot \frac{1}{x_3^{n-1}}} \\
&= \frac{d\left(\frac{x_2/x_3}{x_0/x_3}\right) d\left(\frac{1}{x_0/x_3}\right)}{\frac{\partial F}{\partial X_1}(x_0, x_1, x_2, x_3) \cdot \frac{1}{x_3^{n-1}}} \\
&= \frac{1}{(x_0/x_3)^4} \frac{((x_0/x_3)d(x_2/x_3) - (x_2/x_3)d(x_0/x_3))(-d(x_0/x_3))}{\frac{\partial F}{\partial X_1}(x_0, x_1, x_2, x_3) \cdot \frac{1}{x_3^{n-1}}} \\
&= -\frac{1}{(x_0/x_3)^3} \frac{d(x_2/x_3)d(x_0/x_3)}{\frac{\partial F}{\partial X_1}(x_0, x_1, x_2, x_3) \cdot \frac{1}{x_3^{n-1}}} \\
&= \frac{1}{(x_0/x_3)^3}\sigma_3,
\end{aligned}$$

so that $\sigma_0 = (x_0/x_3)^{n-4}\sigma_3$ in $\Omega^2_{k(X)/k}$. The verification that $\sigma_0 = (x_0/x_2)^{n-4}\sigma_2$ is similar.

To compare σ_0, σ_2, σ_3 with σ_1, recall that

$$\sigma_2 = \frac{d\left(\frac{x_1}{x_2}\right) d\left(\frac{x_3}{x_2}\right)}{\frac{\partial F}{\partial X_0}/x_2^{n-1}}, \quad \sigma_1 = \frac{d\left(\frac{x_3}{x_1}\right) d\left(\frac{x_2}{x_1}\right)}{\frac{\partial F}{\partial X_0}/x_1^{n-1}}.$$

Proceed as above to show that $\sigma_2 = \left(\frac{x_2}{x_1}\right)^{n-4}\sigma_1$. We then have that

$$\sigma_0 = \left(\frac{x_0}{x_3}\right)^{n-4}\sigma_3 = \left(\frac{x_0}{x_2}\right)^{n-4}\sigma_2 = \left(\frac{x_0}{x_2}\right)^{n-4}\left(\frac{x_2}{x_1}\right)^{n-4}\sigma_1 = \left(\frac{x_0}{x_1}\right)^{n-4}\sigma_1.$$

Thus $\sigma_i = \left(\frac{x_i}{x_j}\right)^{n-4}\sigma_j,\ 0 \le i, j \le 3.$ □

Definition 2.19.0. Assume X to be as in (2.17.0). Let $p \in X$ be an isolated singularity, let $A \in k[X]_{(d)}$. Then A is called an *adjoint form of degree d at p* if for every i such that $p \in X_i$ we have A/x_i^d is adjoint at p. (We note that $A/x_i^d \in (\mathcal{O}_x)_p$.)

Definition 2.20.0. $A \in k[X]_{(d)}$ is called an adjoint form if A is adjoint at every singular point of X.

Lemma 2.21.0. $\bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i \cong$ *Adjoint forms of degree* $n-4$ *as* k-*vector spaces.*

Proof. Suppose A is an adjoint form of degree $n-4$ in $k[X]$. We have by (2.18.0)

$$\frac{A}{x_i^{n-4}}\sigma_i = \frac{A}{x_i^{n-4}}\left(\frac{x_i}{x_j}\right)^{n-4}\sigma_j = \frac{A}{x_0^{n-4}}\sigma_0 \qquad \text{for} \qquad 0 \le i,j \le 3.$$

Let $\sigma(A)$ denote the differential in $\Omega^2_{k(X)/k}$ that is equal to

$$\frac{A}{x_0^{n-4}}\sigma_0 = \frac{A}{x_1^{n-4}}\sigma_1 = \frac{A}{x_2^{n-4}}\sigma_2 = \frac{A}{x_3^{n-4}}\sigma_3.$$

Clearly $\sigma(A) \in \bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i$ and σ defines a k-linear injective map from adjoint forms of degree $n-4$ to $\bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i$.

Let $\beta \in \bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i$. Write $\beta = a_0\sigma_0 = a_1\sigma_1 = a_2\sigma_2 = a_3\sigma_3$, $a_i \in \mathrm{Adj}(X_i)$. We have $a_i\sigma_i = a_i(x_i/x_j)^{n-4}\sigma_j = a_j\sigma_j$, so that $a_i(x_i/x_j)^{n-4} = a_j$. This implies the existence of a *unique* form A of degree $n-4$ in $k[X]$ such that $A/x_i^{n-4} = a_i$. Then $\sigma(A) = a_i\sigma_i = \beta$, $i = 0,1,2,3$. Thus σ is surjective and hence a k-vector space isomorphism. □

Lemma 2.22.0. *Let $\widetilde{X} \xrightarrow{\pi} X$ be any desingularization of X. Then π_2: $\Omega^2_{k(\widetilde{X})/k} \to \Omega^2_{k(X)/k}$ defines an isomorphism of k-vector spaces*

$$\pi_2 \colon \omega_{\widetilde{X}}(\widetilde{X}) \xrightarrow{\sim} \bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i.$$

Proof. Let $\widetilde{X}_i = \pi^{-1}(X_i)$. Since π_2 is a monomorphism, the only thing to compute is $\pi_2(\omega_{\widetilde{X}}(\widetilde{X}))$. $\pi_2(\omega_{\widetilde{X}}(\widetilde{X})) = \pi_2(\bigcap_{i=0}^{3}\omega_{\widetilde{X}}(\widetilde{X}_i)) = \bigcap_{i=0}^{3}\pi_2(\omega_{\widetilde{X}}(\widetilde{X}_i)) = \bigcap_{i=0}^{3}(\mathrm{Adj}(X_i))\sigma_i$ (the first equality is by (2.6.1), the second is by the one-to-oneness of π_2, the last equality is by (2.16.0).) □

Theorem 2.23.0. *The following two k-spaces are isomorphic*:

$$\omega_{\widetilde{X}}(\widetilde{X}) \simeq \text{Adjoint forms of degree } n-4 \text{ in } k[X].$$

Proof. By (2.21.0) and (2.22.0). □

Proposition 2.24.0. *Let $X \subseteq \mathbb{P}^3$ be as above. Assume (2.17.1) after renumbering variables. Let $\pi\colon \widetilde{X} \to X$ by any desingularization of X. Let $p \in X$ be an isolated singularity. Let $A \in k[X]_{(d)}$. Let $\widetilde{X}_i = \pi^{-1}(X_i)$. The following conditions are equivalent*:

a) *A is adjoint at p.*

b) *For all* $i = 0, 1, 2, 3$ *such that* $p \in X_i$,

$$\pi_1\left(\frac{A}{x_i^d}\sigma_i\right) \in Reg_{\widetilde{X}}(\pi^{-1}(p)).$$

c) *For some* $i = 0, 1, 2, 3$ *such that* $p \in X_i$,

$$\pi_1\left(\frac{A}{x_i^d}\sigma_i\right) \in Reg_{\widetilde{X}}(\pi^{-1}(p)).$$

d) *For some* $i = 0, 1, 2, 3$ *such that* $p \in X_i$, A/x_i^d *is adjoint at* p *as an element of* $(\mathcal{O}_{x_i})_p$.

Corollary 2.24.1. *Retain the notations of (2.24.0). Assume* $p \in X_0$; *then* A *is adjoint at* p *if and only if* $A(1, y_1, y_2, y_3)$ *is adjoint at the point of*

$$H_0 = Spec\,\frac{k[y_1, y_2, y_3]}{(F(1, y_1, y_2, y_3))}$$

that corresponds to p. *Similarly if* $p \in X_1, X_2$, *or* X_3, *with* H_0 *replaced by* H_1, H_2, *or* H_3, *and* A *suitable dehomogenized.*

Remark 2.24.2. (2.24.1) means that in order to check whether A is an adjoint form we can dehomogenize the equation of X and A and reduce to an affine hypersurface case.

Proof of Proposition 2.24.0. Let $\widetilde{X}_i = \pi^{-1}(X_i)$.

(a)⇒(b): Assume (a). For every i such that $p \in X_i$,

$$\pi_1\left(\frac{A}{x_i^d}\right) \in \mathrm{Reg}_{\widetilde{X}_i}(\pi^{-1}(p)) = \mathrm{Reg}_{\widetilde{X}}(\pi^{-1}(p)).$$

(b)⇒(a): Also follows since $\mathrm{Reg}_{\widetilde{X}_i}(\pi^{-1}(p)) = \mathrm{Reg}_{\widetilde{X}}(\pi^{-1}(p))$.

(b)⇒(c): Obvious.

(c)⇒(d): Follows trivially from the definition of adjoints.

(c)⇒(b): Assume (c). Let $j \neq i$, $p \in X_j$,

$$\begin{aligned}\pi_1\left(\frac{A}{x_j^d}\sigma_j\right) &= \pi_1\left(\left(\frac{x_j}{x_i}\right)^{n-4}\frac{A}{x_j^d}\sigma_i\right)\\ &= \pi_1\left(\left(\frac{x_j}{x_i}\right)^{n-4-d}\left(\frac{A}{x_i^d}\right)\sigma_i\right)\\ &= \pi_1\left(\left(\frac{x_j}{x_i}\right)^{n-4-d}\right)\cdot\pi_1\left(\frac{A}{x_i^d}\sigma_i\right)\end{aligned}$$

Since we assume (c), it is enough by (2.5.0) to see that $\pi_1((x_j/x_i)^{n-4-d})$ is regular on $\pi^{-1}(p)$. But $(x_j/x_i)^{n-4-d} \in (\mathcal{O}_x)_p$ since $p \in X_i \cap X_j$, so (b) follows. $\square$

Remark 2.24.3. If g is any form of degree 1 such that $g(p) \neq 0$, A is adjoint at p if and only if A/g^d is adjoint to X_g. Hence the definition of adjoint forms is invariant under a linear change of coordinates.

Theory of ℓ-Adjoints for $\ell \geq 1$ (An Outline)

We shall use notations from the theory of adjoints. We consider $\Omega^{2\otimes\ell}_{k(X)/k}$, $\otimes$ over $k(X)$. We define $\ell - \mathrm{Reg}_X(p) = \{$elements of $\Omega^{2\otimes\ell}_{k(X)/k}$ of the form $\Sigma c_i\mu_i$, where $\mu_i = (da_{i1}db_{i1}) \otimes (da_{i2}db_{i2}) \otimes \cdots \otimes (da_{i\ell}db_{i\ell})$ and $c_i, a_{ij}, b_{ij} \in \mathcal{O}_p\}$. We define for each $\ell \geq 1$ a sheaf $\ell - \omega_X(U) = \bigcap_{q\in U'} \ell - \mathrm{Reg}_X(p)$.

Proposition 2.25.0. *$\ell - \omega_X(U)$ is an $\mathcal{O}_x$-module.*

Proposition 2.26.0. *$\ell - \omega_X(U)$ is canonically isomorphic to $\omega_X^{\otimes\ell}$. From now on $\ell - \omega_X$ and ω_X^ℓ, and $\ell - Reg_X(p)$, $Reg_x^{\otimes\ell}(p)$ will be used interchangeably.*

$\omega_X^{\otimes\ell}$ has an analogous theory to ω_X. Propositions (2.3.0) - (2.8.9) have obvious analogues for $\omega_X^{\otimes\ell}$. In particular, we note for $F \subseteq A^3$ as in (2.8.3) and (2.9.0) we have for $U \subseteq F$:

Proposition 2.27.0. $\omega_F^{\otimes\ell}(U) = \mathcal{O}_F(U)\sigma_F^{\otimes\ell}$.

Definition 2.28.0. Let $F \subseteq A^3$ and $p \in F$ be as in (2.13.0), $\widetilde{F} \to F$ a resolution of p. An element $a \in (\mathcal{O}_F)_p$ is said to be *ℓ-adjoint* if and only if $\pi_1(a\sigma_F^{\otimes\ell}) = \pi_1(a) \cdot \pi_1(\sigma_F^{\otimes\ell}) = \pi_1(a)\ (\pi_1(\sigma_F))^{\otimes\ell}$ is regular along $\pi^{-1}(p)$.

Again Propositions (2.13.0) - (2.16.0) have obvious analogies for ℓ-adjoints. A new fact is the following.

Lemma 2.28.1. *Let $p \in F \subseteq A^3$ be as in (2.19.0). If a_1 is ℓ_1-adjoint and a_2 is ℓ_2-adjoint at p, then a_1a_2 is $\ell_1 + \ell_2$ adjoint at p.*

Proof. Immediate from the definition.

Passing to the projective case, let $X \subseteq P^3$ be as in (2.17.0).

Definition 2.29.0. $A \in k[X]_{(d)}$ is *ℓ-adjoint* at $p \in X$ if A/x_i^d is ℓ-adjoint on the scheme X_i at p for each i such that $p \in X_i$.

Definition 2.30.0. A is *ℓ-adjoint* to X if A is ℓ-adjoint at all singular points of X.

Again we obtian propositions analogous to (2.24.0) - (2.24.3). The key difference is the following.

Remark 2.31.0. $\sigma_1^{\otimes\ell} = (x_i/x_j)^{\ell(n-4)}\sigma_j^{\otimes\ell}$, which is a consequence of (2.18.0).

Exactly as in the theory of adjoints we prove:

Theorem 2.32.0. *$H^0(\widetilde{X}, \omega_{\widetilde{X}}^{\otimes \ell})$ is isomorphic as a k-vector space to the space of ℓ-adjoint forms of degree $\ell(n-4)$.*

Proposition 2.33.0. *Let A_1 be an ℓ_1-adjoint form of degree d_1 (at p), A_2 be an ℓ_2-adjoint form of degree d_2 (at p). Then A_1A_2 is an $\ell_1+\ell_2$ adjoint form (at p) of degree d_1+d_2.*

Valuation Theory for Differentials

We need only deal with the case of a normal hypersurface $F \subseteq A^3$, $F = \operatorname{Spec}(R)$ where $R = k[x,y,z]/(f(x,y,z))$. Choose a canonical generating differential σ_F. Let v be a discrete valuation of $k(F)$ that corresponds to a height one prime ideal of R. We extend v to $\Omega^2_{k(F)/k}$ by setting $v(\alpha) = v(a)$ where $\alpha = a\sigma_F$.

This is clearly independent of the choices of σ_F as v is equal to 0 on units of R.

Proposition 2.34.1. *$\alpha \in \Omega^2_{k(F)/k}$ is regular on F if and only if for every valuation v corresponding to a height one prime of R, $v(\alpha) \geq 0$.*

Proof. Set $\alpha = a\sigma_F$. α is regular on F if and only if $a \in R$; that is, if and only if $v(a) \geq 0$ for all v as in the proposition; but $v(\alpha) = v(a)$ for all such valuations by definition. □

Lemma 2.35.0. *Let X be a smooth surface. Then any point $p \in X$ has an affine neighborhood $U(p)$ with the following properties:*

(a) *$U(p) = Spec(R)$, $R \subseteq k(X)$, R has fraction field $k(X)$; R is regular.*

(b) *$\omega_{X|U(p)}$ is free on one generator $\sigma_R \in \Omega^2_{k(X)/k}$; i.e., if $V \subseteq U$ then $\omega_X(V) = \mathcal{O}_X(V)\sigma_R$ is a free $\mathcal{O}_X(V)$-module on one generator. In particular, $\omega_X(U(p)) = R\sigma_R$.*

Proof. In this case ω_X is the sheaf of Kähler differentials and the result is well known; for example see [12], p. 335.

Propositon 2.36.0. *Let F be isomorphic to a hypersurface in $\mathbb{A}^3$; F is normal. Let $p \in F$ be an isolated singularity. Then the ideal of adjoints $\operatorname{Adj}(p) \subseteq (\mathcal{O}_F)_p$ is integrally closed.*

Proof. Let $\pi\colon \widetilde{F} \to F$ be a desingularization. Suppose $\operatorname{Adj}(p)$ is not integrally closed. Then for some $r \in (\mathcal{O}_F)_p - \operatorname{Adj}(p)$,

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0, \tag{1}$$

where $a_i \in [\operatorname{Adj}(p)]^i$, $1 \leq i \leq n$. Let σ_F be a canonical generating differential on F. $\pi_1(r\sigma_F)$ fails to be regular at some point $q \in \pi^{-1}(p)$. Choose a neighborhood $U(q) = \operatorname{Spec}(R)$ of q in $\widetilde{F}$ with $R \subseteq k(\widetilde{F})$ as in (2.35.0). Then

$$\pi_1(\sigma_F) = \gamma\sigma_R, \qquad \text{where} \qquad \gamma \in k(\widetilde{F}),$$
$$\pi_1(r\sigma_F) = \pi_1(r)\cdot\gamma\sigma_R,$$

so

$$\pi_1(r) \cdot \gamma \notin R.$$

On the other hand, if $a \in \mathrm{Adj}(p)$, then $\pi_1(a) \cdot \gamma \in R$. Since R is normal, there exists a discrete valuation v such that

$$v(\pi_1(r) \cdot \gamma) = v(\pi_1(r)) + v(\gamma) < 0 \qquad (2)$$

whereas $v(\pi_1(a)) + v(\gamma) \geq 0$ if $a \in \mathrm{Adj}(p)$.

If $a_i \in [\mathrm{Adj}(p)]^i$,

$$v(\pi_1(a_i)) + i\, v(\gamma) \geq 0. \qquad (3)$$

We prove that (1), (2), (3) are incompatible. Apply π_1 to (1) to obtain

$$(\pi_1(r))^n = -\pi_1(a_1)(\pi_1(r))^{n-1} - \pi_1(a_2)(\pi_1(r))^{n-2} \ldots - \pi_1(a_n).$$

Applying v to both sides yields

$$n\, v(\pi_1(r)) \geq v(\pi_1(a_{i_0}) \cdot (\pi_1(r))^{n-i_0}) = v(\pi_1(a_{i_0})) + (n - i_0) v(\pi_1(r))$$

for some

$$1 \leq i_0 \leq n; \qquad \text{so} \qquad i_0\, v(\pi_1(r)) \geq v(\pi_1(a_{i_0})).$$

By (3), $v(\pi_1(a_{i_0})) \geq -i_0\, v(\gamma)$. So $i_0\, v(\pi_1(r)) \geq -i_0\, v(\gamma)$ or $v(\pi_1(r)) \geq -v(\gamma)$, so $v(\pi_1(r)) + v(\Gamma) \geq 0$ which contradicts (2). $\square$

Remark 2.37.0. Similarly, we can extend valuation theory to $(\Omega^2_{k(F)/k})^{\otimes 2}$ by setting $v(a\sigma_F^{\otimes 2}) = v(a)$. For $\alpha \in (\Omega^2_{k(F)/k})^{\otimes 2}$ we again have that α is regular on F if and only if $v(\alpha) \geq 0$ for all valuations as in Proposition 2.34.1.

References

1. S. S. Abhyankar, *Local uniformation on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math. 63 (19), pp. 491–526.

2. M. Artin, *On isolated rational singularities of surfaces,* Amer. J. Math. 88 (1966), p. 129.

3. P. Blass, *Zariski Surfaces,* Thesis, University of Michigan, 1977.

4. P. Blass and J. Lipman, *Remarks on adjoints and arithmetic genera of algebraic varieties,* Amer. J. Math. 101 (2) (1979), pp. 331–336.

5. F. Enriques, *Le superficie Algebriche,* Nicola Zanichelli Editore, Bologna, 1949.

6. H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologie-grupen auf komplexen Räumen,* Inventiones Math. 11 (4) (1970), p. 271.

7. S. Lang, *Abelian Varieties,* Interscience Tracts, in *Pure and Applied Mathematics,* No. 7, New York.

8. W. E. Lang, *quasi-elliptic surfaces in characteristic three,* Thesis, Harvard University, 1978.

9. J. Lipman, *Rational Singularities,* IHES Publications, No. 36.

10. J. S. Milne, *Étale cohomology,* Princeton University Press, 1966.

11. D. Mumphord, *Lectures on curves on an algebraic surface,* Ann. of Math. Studies, No. 59, Princeton University Press, 1966.

12. D. Mumphord, *Introduction to Algebraic Geometry,* Harvard Lecture Notes, p. 335.

13. M. Nagata, *Local rings,* Interscience Tracts, in *Pure and Applied Mathematics* 13.

14. Y. Nakai, *On the characteristic linear system of algebraic families, III,* J. Math. I (1957), pp. 552–561.

15. I. Safarevic, *Minimal Models,* Tata Institute of Fundmental Research, Bombay, 1966.

16. I. Safarevic, *Basic Algebraic Geometry,* Springer, 1974.

17. J. P. Serre, *Faisceaux algébrique coherent,* Ann. of Math. 61 (1955), pp. 197–278.

18. O. Zariski, *Introduction to the problem of minimal models in the theory of algebraic surfaces,* Publ. Math. Soc. Japan 4 (1958).

19. O. Zariski, *The problem of minimal models in the theory of algebraic surfaces,* Amer. J. of Math. 80 (1958).

20. O. Zariski, *On Castelnuovo's Criterion of Rationality* $p_a = P_2 = 0$ *of an algebraic surface, III,* J. of Math. 2 (3) (1958).