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On the Polynomial Representation of Certain Recurrences

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Abstract

Any polynomial $P_r(n)$ of degree r in the integral indeterminate n is given by the n-th element T_n of the sequence $\{T_n\}$ obeying the homogeneous linear recurrence $T_n = \binom{R}{1}T_{n-1} - \binom{R}{2}T_{n-2} + \cdots + (-1)^{R-1}\binom{R}{R}T_{n-R}$ $(n \geq R)$ of order $R \geq r+1$. Once the polynomial is given, the initial conditions of the recurrence are easily found to be $T_i = P_r(i)$ $(0 \leq i \leq R-1)$. In this note we face the inverse problem, namely, for given initial conditions T_i, we find explicit expressions for the coefficients c_i of the polynomial $T_n = P_r(n) = c_r n^r + c_{r-1}n^{r-1} + \cdots + c_1 n + c_0$ of degree $r = R-1$ which is associated to the above recurrence. Simplified expressions of c_i are given for particular values of i and closed-form expressions of these coefficients are established for the first few values of r. Finally, some combinatorial expressions are shown which emerge from certain special cases of the obtained results.

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1. Introduction

My wife is a housewife having a bent for mathematics. She sometimes amuses herself by posing and solving geometrical problems. Some months ago, she decided to determine the number d_n of diagonals of a polygon with n sides and found, empirically, the following expressions

$$d_n = n(n-3)/2, \tag{1.1}$$

$$d_n = d_{n-1} + n - 2, [d_3 = 0], \tag{1.2}$$

$$d_n = 3d_{n-1} - 3d_{n-2} + d_{n-3}, [d_3 = 0, d_4 = 2, d_5 = 5]. \tag{1.3}$$

She asked me how it is possible that three distinct relations yield the same (correct) result. I could not give an immediate answer. As a matter of fact, the general answer to this kind of question led me to write the survey paper [2] and this note.

It can be readily proved (e.g., see [2]) that any polynomial $P_r(n)$ of degree r in the integral indeterminate n is given by the n-th element T_n of the sequence obeying the *homogeneous* linear recurrence of order $R \geq r+1$

$$T_n = \sum_{j=1}^{R} (-1)^{j-1} \binom{R}{j} T_{n-j} \quad (n \geq R). \tag{1.4}$$

In fact, the characteristic equation associated to (1.4)

$$(1-z)^R = 0 \tag{1.5}$$

has the root 1 with multiplicity R so that the solution of (1.4) has the form

$$T_n = \sum_{k=0}^{R-1} A_k n^k, \tag{1.6}$$

where the coefficients A_k are to be determined on the basis of the R initial conditions $T_i (0 \leq i \leq R-1)$ of (1.4). Obviously, we have

$$T_i = P_r(i) \quad (0 \leq i \leq R-1). \tag{1.7}$$

It must be noted that the polynomial $P_r(n)$ can also be given by the element Q_n of a sequence obeying a linear recurrence of order $S < R+1$. In this case, the recurrence relation is *not* homogeneous [2] and has the form

$$Q_n = \sum_{j=1}^{S} (-1)^{j-1} \binom{S}{j} Q_{n-j} + f(n) \quad (n \geq S), \tag{1.8}$$

where $f(n)$ is a polynomial in n of degree less than or equal to r (cf. (1.2)). As an example, let us consider the polynomial

$$P_3(n) = 2n^3 + 5n^2 - 3n + 4. \tag{1.9}$$

Some possible recurrences for (1.9) are

$$T_n = \sum_{j=1}^{5} (-1)^{j-1} \binom{5}{j} T_{n-j} \quad (n \geq 5) \tag{1.10}$$

with initial conditions $T_0 = 4$, $T_1 = 8$, $T_2 = 34$, $T_3 = 94$, $T_4 = 200$,

$$T_n = \sum_{j=1}^{4} (-1)^{j-1} \binom{4}{j} T_{n-j} \quad (n \geq 4) \tag{1.11}$$

with initial conditions $T_0 = 4$, $T_1 = 8$, $T_2 = 34$, $T_3 = 94$,

$$T_n = \sum_{j=1}^{3} (-1)^{j-1} \binom{3}{j} T_{n-j} + 12 \quad (n \geq 3) \tag{1.12}$$

with initial conditions $T_0 = 4$, $T_1 = 8$, $T_2 = 34$, and

$$T_n = \sum_{j=1}^{2} (-1)^{j-1} \binom{2}{j} T_{n-j} + 12n - 2 \quad (n \geq 2) \tag{1.13}$$

with initial conditions $T_0 = 4$, $T_1 = 8$.
Observe that, in the above examples, the minimum admissible value of R for the recurrence to be homogeneous is $R = r + 1 = 4$ (see (1.11)).

As said before, if $P_r(n)$ is given, the initial conditions of (1.4) are given by (1.7). In this note we face the inverse problem: once the recurrence of (1.4) and the initial conditions T_i $(0 \leq i \leq R-1)$ are given, find explicit expressions for the coefficients c_i $(0 \leq i \leq R-1)$ of the polynomial

$$T_n = P_r(n) = c_r n^r + c_{r-1} n^{r-1} + \cdots + c_1 n + c_0 \tag{1.14}$$

of degree $r = R - 1$, associated to (1.4), in terms of the initial conditions T_i (Section 2). Observe that, depending on the choice of T_i, some of the c_i may vanish so that the real degree of the polynomial may be less than r (see Section 5.1). In Section 3 simplified expressions for c_i are established for particular values of i whereas closed-form expressions for them are given in Section 4, for the first few values of r. Finally, in Section 5, some polynomials of degree less than r are shown which emerge from particular choices of T_i. Moreover, it is shown how some combinatorial identities can be found by using the results established in the previous sections.

2. Determination of the Coefficients c_i

From (1.14) we immediately obtain

$$c_0 = T_0. \tag{2.1}$$

Furthermore, for $1 \leq i \leq r$, we can write the system of r equations in the r unknowns c_i

$$\begin{cases} c_1 + c_2 + \cdots + c_r = T_1 - T_0 \\ 2c_1 + 2^2 c_2 + \cdots + 2^r c_r = T_2 - T_0 \\ \vdots \\ r c_1 + r^2 c_2 + \cdots + r^r c_r = T_r - T_0 \end{cases} \tag{2.2}$$

that is

$$\begin{cases} c_1 + c_2 + \cdots + c_r = T_1 - T_0 \\ c_1 + 2c_2 + \cdots + 2^{r-1}c_r = (T_2 - T_0)/2 \\ \vdots \\ c_1 + rc_2 + \cdots r^{r-1}c_r = (T_r - T_0)/r \end{cases} . \tag{2.3}$$

In matrix form, we require the solution $\mathbf{c}$ of the equation

$$\mathbf{M}\,\mathbf{c} = \mathbf{t}, \tag{2.4}$$

where $\mathbf{M}$ is the particular r-by-r *Vandermonde matrix*

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{r-1} \\ \vdots & & & & \\ 1 & r & r^2 & \dots & r^{r-1} \end{bmatrix},$$

and

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix}, \mathbf{t} = \begin{bmatrix} T_1 - T_0 \\ (T_2 - T_0)/2 \\ \vdots \\ (T_r - T_0)/r \end{bmatrix}.$$

The solution of (2.3) is clearly

$$\mathbf{c} = \mathbf{M}^{-1}\mathbf{t}. \tag{2.7}$$

Let $\mathbf{M}^{-1} = [\nu_{ij}(r)]$. The entries $\nu_{ij}(r)$ can be obtained by particularizing to $\mathbf{M}$ the explicit expression for the entries of the inverse of a generic Vandermonde matrix [4, pp. 27-28]. Namely, we have

$$\begin{aligned} \nu_{ij}(r) = j\frac{(-1)^{r-i}\sigma^{(j)}_{r-i}(r)}{\prod\limits_{\substack{k=0 \\ k\neq j}}^{r}(j-k)} &= \frac{(-1)^{i+j}\sigma^{(j)}_{r-i}(r)}{(j-1)!(r-j)!} \\ &= (-1)^{i+j}\frac{j}{r!}\binom{r}{j}\sigma^{(j)}_{r-i}(r), \end{aligned} \tag{2.8}$$

where the combinatorial entity $\sigma^{(j)}_m(r)$ denotes the sum of all products of m of the integers $\{1, 2, \dots, j-1, j+1, \dots, r-1, r\}$ without permutations or repetitions. For example, we have $\sigma^{(4)}_3(6) = (1\cdot 2\cdot 3) + (1\cdot 2\cdot 5) + (1\cdot 2\cdot 6) +$

$(1\cdot3\cdot5)+(1\cdot3\cdot6)+(1\cdot5\cdot6)+(2\cdot3\cdot5)+(2\cdot3\cdot6)+(2\cdot5\cdot6)+(3\cdot5\cdot6)=307.$
Special cases for $\sigma_m^{(j)}(r)$ are

$$\sigma_0^{(j)}(r) \overset{\text{def}}{=} 1, \tag{2.9}$$

$$\sigma_1^{(j)}(r) = \frac{r^2+r}{2} - j = \binom{r+1}{2} - j \quad (r \geq 2), \tag{2.10}$$

$$\sigma_2^{(j)}(r) = \binom{r}{2}\frac{3r^2+5r+2}{12} - j\left[\binom{r+1}{2} - j\right] \ (r \geq 3), \tag{2.11}$$

$$\sigma_{r-1}^{(j)}(r) = r!/j \quad (r \geq 1). \tag{2.12}$$

The proofs of (2.10) and (2.12) are immediate, whereas the proof of (2.11) is rather tedious and is omitted for brevity. Let us observe *en passant* that, on the basis of [6, p. 98], the quantity $\sigma_m^{(j)}(r)$ can be expressed in terms of Stirling numbers of the first kind $S_k^{(i)}$. Namely, we have

$$\sigma_{r-i}^{(j)} = (r-j)!\sum_{k=i}^{r}(-1)^{k+i}\frac{S_k^{(i)}}{(k-j)!}. \tag{2.13}$$

It has to be noted that all the addends in the sum (2.13) are positive and this expression holds for all j provided the convention $1/(-|k|)! = 0$ (see the definition of the Γ function) is assumed. For $i = r$, (2.13) reduces to

$$\sigma_0^{(j)}(r) = (r-j)!(-1)^{2r}S_r^{(r)}/(r-j)! = 1, \tag{2.14}$$

which is consistent with the definition (2.9).

Getting back to the point, from (2.7) we can write

$$\begin{aligned} c_i &= \nu_{i1}(r)(T_1 - T_0) + \nu_{i2}(r)\frac{T_2 - T_0}{2} + \cdots + \nu_{ir}(r)\frac{T_r - T_0}{r} \\ &= -T_0\sum_{j=1}^{r}\frac{\nu_{ij}(r)}{j} + \sum_{j=1}^{r}\frac{\nu_{ij}(r)}{j}T_j. \end{aligned} \tag{2.15}$$

Finally, from (2.15) and (2.8), after some simple manipulations we get the following expression valid for $1 \leq i \leq r$

$$\begin{aligned} c_i = \frac{(-1)^{i+1}}{r!}\Bigg[T_0\sum_{j=1}^{r}(-1)^j\binom{r}{j}\sigma_{r-i}^{(j)}(r) \\ - \sum_{j=1}^{r}(-1)^j\binom{r}{j}\sigma_{r-i}^{(j)}(r)T_j\Bigg]. \end{aligned} \tag{2.16}$$

The formulae (2.1) and (2.16) give the solution of the problem.

3. The Coefficients c_i for Particular Values of i

For particular values of i, the expression (2.16) simplifies remarkably. Let us examine some cases.

The coefficient c_1 ($r \geq 1$)

Letting $i = 1$ in (2.16) and using (2.12) yield

$$c_1 = T_0 \sum_{j=1}^{r} (-1)^j \binom{r}{j} \frac{1}{j} - \sum_{j=1}^{r} (-1)^j \binom{r}{j} \frac{T_j}{j}. \tag{3.1}$$

Denoting the r-th *harmonic number* [5, p. 73] by H_r,

$$H_r = \sum_{i=1}^{r} 1/i, \tag{3.2}$$

and using the noteworthy identity available in [7, Ex. 3, pp. 4-5], the expression (3.1) can be rewritten as

$$c_1 = -H_r T_0 - \sum_{j=1}^{r} (-1)^j \binom{r}{j} \frac{T_j}{j}. \tag{3.3}$$

The coefficient c_r ($r \geq 1$)

Letting $i = r$ in (2.16) and using (2.9) yield

$$c_r = \frac{(-1)^{r+1}}{r!} \left[T_0 \sum_{j=1}^{r} (-1)^j \binom{r}{j} - \sum_{j=1}^{r} (-1)^j \binom{r}{j} T_j \right]. \tag{3.4}$$

Since

$$\sum_{j=1}^{r} (-1)^j \binom{r}{j} = -1 \quad (r \geq 1), \tag{3.5}$$

the expression (3.4) can be rewritten as

$$c_r = \frac{(-1)^{r+1}}{r!} \left[-T_0 - \sum_{j=1}^{r} (-1)^j \binom{r}{j} T_j \right] = \frac{(-1)^r}{r!} \sum_{j=0}^{r} (-1)^j \binom{r}{j} T_j. \tag{3.6}$$

The coefficient c_{r-1} ($r \geq 2$)

Letting $i = r - 1$ in (2.16) and using (2.10) yield

$$\begin{aligned} c_{r-1} = \frac{(-1)^r}{r!} \Bigg\{ & T_0 \binom{r+1}{2} \sum_{j=1}^{r} (-1)^j \binom{r}{j} - T_0 \sum_{j=1}^{r} (-1)^j \binom{r}{j} j \\ & - \sum_{j=1}^{r} (-1)^j \binom{r}{j} \left[\binom{r+1}{2} - j \right] T_j \Bigg\}. \end{aligned} \tag{3.7}$$

Now let us observe that

$$\sum_{j=1}^{r}(-1)^j\binom{r}{j}j^k = 0 \quad \text{if} \quad r > k > 0. \tag{3.8}$$

The proof of (3.8) can be carried out by taking the successive derivatives (with respect to x) of both $(1-x)^r$ and the corresponding binomial coefficient expansion, setting $x = 1$ and using induction on k. On the other hand, (3.8) results from the definition and the closed-form expression of the Stirling numbers of the second kind (e.g., see [1, p. 824]).

Replacing (3.5) and (3.8) (with $k = 1$) in (3.7) yields

$$c_{r-1} = \frac{(-1)^r}{r!}\left\{-\binom{r+1}{2}T_0 - \sum_{j=1}^{r}(-1)^j\binom{r}{j}\left[\binom{r+1}{2} - j\right]T_j\right\}. \tag{3.9}$$

The coefficient c_{r-2} ($r \geq 3$)

From (2.16) and (2.11), by using (3.5) and (3.8) (with $k = 1$ and 2) we get, after some simple manipulations

$$\begin{aligned} c_{r-2} = \frac{(-1)^{r-1}}{r!}\Bigg\{&-\binom{r}{2}N_r T_0 \\ &- \sum_{j=1}^{r}(-1)^j\binom{r}{j}\left[\binom{r}{2}N_r - j\binom{r+1}{2} + j^2\right]T_j\Bigg\}, \end{aligned} \tag{3.10}$$

where (see (2.11)) $N_r = (3r^2 + 5r + 2)/12$.

4. The Coefficients c_i for the First Few Values of r

As an illustration, we give the expressions of the coefficients c_i in terms of the initial conditions T_i $(0 \leq i \leq r)$ for $0 \leq r \leq 5$. These expressions can be derived from (2.1) and (2.16), after a good deal of calculation.

(i) $r = 0$

$$c_0 = T_0.$$

(ii) $r = 1$

$$c_0 = T_0, \; c_1 = -T_0 + T_1.$$

(iii) $r = 2$

$$c_0 = T_0, \; c_1 = \frac{-3T_0 + 4T_1 - T_2}{2}, \; c_2 = \frac{T_0 - 2T_1 + T_2}{2}.$$

(iv) $r = 3$

$$c_0 = T_0, \qquad c_1 = \frac{-11T_0 + 18T_1 - 9T_2 + 2T_3}{6},$$
$$c_2 = \frac{2T_0 - 5T_1 + 4T_2 - T_3}{2}, \quad c_3 = \frac{-T_0 + 3T_1 - 3T_2 + T_3}{6}.$$

(v) $r = 4$

$$c_0 = T_0,\ c_1 = \frac{-25T_0 + 48T_1 - 36T_2 + 16T_3 - 3T_4}{12},$$
$$c_2 = \frac{35T_0 - 104T_1 + 114T_2 - 56T_3 + 11T_4}{24},$$
$$c_3 = \frac{-5T_0 + 18T_1 - 24T_2 + 14T_3 - 3T_4}{12},\ c_4 = \frac{T_0 - 4T_1 + 6T_2 - 4T_3 + T_4}{24}.$$

(vi) $r = 5$

$$c_0 = T_0,\ c_1 = \frac{-137T_0 + 300T_1 - 300T_2 + 200T_3 - 75T_4 + 12T_5}{60},$$
$$c_2 = \frac{45T_0 - 154T_1 + 214T_2 - 156T_3 + 61T_4 - 10T_5}{24},$$
$$c_3 = \frac{-17T_0 + 71T_1 - 118T_2 + 98T_3 - 41T_4 + 7T_5}{24},$$
$$c_4 = \frac{3T_0 - 14T_1 + 26T_2 - 24T_3 + 11T_4 - 2T_5}{24},$$
$$c_5 = \frac{-T_0 + 5T_1 - 10T_2 + 10T_3 - 5T_4 + T_5}{120}.$$

The above expressions can be readily checked on against (3.3), (3.6), (3.9) and (3.10), under the appropriate restrictions on r.

Remark

Even though we restrict the choice of the initial conditions T_i to integers, the coefficients c_i are not, in general, integers. The integrality of *all* the c_i can emerge from particular integral values of T_i. From (i)–(vi) it can be readily seen that, for $r = 0$ and 1, all the c_i are integers, for $r = 2$, all the c_i are integers iff T_0 and T_2 have the same parity whereas, for $r = 3$, all the c_i are integers iff $T_0 \equiv T_3 \pmod 3$, $T_1 \equiv T_3 \pmod 2$, and $T_2 \equiv T_0 \pmod 2$. For example, letting $T_0 = 11$, $T_1 = 10$, $T_2 = 5$ and $T_3 = 32$ in (iv), we get $c_0 = 11$, $c_1 = 13$, $c_2 = -20$ and $c_3 = 6$.

5. Special Cases

Particular choices of the initial conditions T_i lead to particular polynomials $P_r(n)$ whereas, with the aid of (1.7), (2.15) and (2.8), particular choices of the coefficients c_i give rise to some interesting combinatorial identities.

5.1 Particular choices of the initial conditions

(i) $T_j = (-1)^{r+1} T_{r-j}$ $(0 \le j \le r)$

From (3.6), we have $c_r = 0$ (i.e., the degree of the polynomial is $r-1$). Observe that (i) implies that $T_{r/2} = 0$ if r is even.

(ii) $T_j = j$ $(0 \le j \le r)$

From (2.1) we have $c_0 = 0$ and, from (2.15), we have

$$c_i = \sum_{j=1}^{r} \nu_{ij} \qquad (1 \le i \le r). \tag{5.1}$$

Now, observing that $[\nu_{ij}]\mathbf{M} = \mathbf{I}$ by definition ($\mathbf{I}$ being the r-by-r identity matrix) and that the first column of $\mathbf{M}$ is a unit vector, we can write

$$\sum_{j=1}^{r} \nu_{ij} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } 2 \le i \le r. \end{cases} \tag{5.2}$$

From (5.1) and (5.2), it is evident that $P_r(n) = P_1(n) = n$.

(iii) $T_j = 1$ $(0 \le j \le r)$

From (2.1), we have $c_0 = 1$ and, from (2.15), $c_i = 0$ $(1 \le i \le r)$. It follows that $P_r(n) = P_0(n) = 1$.

5.2 Particular choices of the coefficients

Let us suppose that the coefficients c_i of $P_r(n)$ are given, with $c_0 = 0$. From (1.7) we have

$$T_j = \sum_{i=1}^{r} c_i j^i, \tag{5.3}$$

whence using (2.16) and taking into account that $T_0 = c_0 = 0$ by hypothesis yield

$$\sum_{j=1}^{r} (-1)^j \binom{r}{j} \sigma_{r-i}^{(j)}(r) T_j = (-1)^i c_i r!. \tag{5.4}$$

For particular choices of c_i, the quantity T_j given by the sum (5.3) has a rather compact closed-form expression so that (5.4) becomes a compact identity involving the quantities $\sigma_m^{(j)}(r)$ (or the Stirling numbers of the first kind (see (2.13)). Specializing i to r and 1 (see (2.9) and (2.12)) on the left-hand side of (5.4) leads to some interesting combinatorial identities. We give four examples by letting $c_i = 1$, $c_i = i$, $c_i = \binom{r}{i}$ and $c_i = \binom{r-i}{i}$ $(1 \le i \le r)$ on the right-hand side of (5.4).

(i) $c_i = 1$

From (5.3) we have

$$\begin{cases} T_1 = r \\ T_j = j(j^r - 1)/(j-1) \quad \text{for } 2 \le j \le r. \end{cases} \tag{5.5}$$

From (5.4) and (5.5) we can write

$$-r^2 \sigma^{(1)}_{r-i}(r) + \sum_{j=2}^{r} (-1)^j j \frac{j^r - 1}{j-1} \binom{r}{j} \sigma^{(j)}_{r-i}(r) = (-1)^i r!. \tag{5.6}$$

Letting $i = r$ and $i = 1$ in (5.6), from (2.9) and (2.12), we obtain

$$\sum_{j=2}^{r} (-1)^j j \frac{(j^r - 1)}{j-1} \binom{r}{j} = (-1)^r r! + r^2 \tag{5.7}$$

and

$$\sum_{j=2}^{r} (-1)^j \frac{j^r - 1}{j-1} \binom{r}{j} = r^2 - 1, \tag{5.8}$$

respectively.

By means of an analogous procedure, we obtained the further results shown in the sequel.

(ii) $c_i = i$

$$\sum_{j=2}^{r} (-1)^j j \frac{j^r (jr - r - 1) + 1}{(j-1)^2} \binom{r}{j} = r \left[(-1)^r r! + \binom{r+1}{2} \right], \tag{5.9}$$

$$\sum_{j=2}^{r} (-1)^j \frac{j^r (jr - r - 1) + 1}{(j-1)^2} \binom{r}{j} = r \binom{r+1}{2} - 1. \tag{5.10}$$

(iii) $c_i = \binom{r}{i}$

$$\sum_{j=1}^{r} (-1)^j [(j+1)^r - 1] \binom{r}{j} = (-1)^r r!, \tag{5.11}$$

$$\sum_{j=1}^{r} (-1)^j \frac{(j+1)^r - 1}{j} \binom{r}{j} = -r. \tag{5.12}$$

(iv) $c_i = \binom{r-i}{i}$ ($c_i = 0$ if $i > \lfloor r/2 \rfloor$)

$$\sum_{j=1}^{r} (-1)^j \binom{r}{j} [G_{r+1}(j) - 1] = 0, \tag{5.13}$$

$$\sum_{j=1}^{r} (-1)^j \frac{1}{j} \binom{r}{j} [G_{r+1}(j) - 1] = 1 - r, \tag{5.14}$$

where the numbers $G_n(m)$ can be defined either as (e.g., see [3])

$$G_{n+2}(m) = G_{n+1}(m) + mG_n(m) \; [G_0(m) = 0, \, G_1(m) = 1] \tag{5.15}$$

or as (e.g., see [7, p. 75])

$$G_n(m) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} m^k. \tag{5.16}$$

The combinatorial identities obtainable in this way are by no means exhausted in our brief account above. For example, the following question arises quite naturally for those interested in recurring sequences: "Which couple of identities shall we obtain if we let c_i be the i-th Fibonacci number F_i?" We leave the answer as an exercise for the interested reader.

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