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Alexandre Grothendieck's EGA V
Part VI
(Interpretation and Rendition of his 'prenotes')

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§**14 Conic Projections** N.B. We have already used Conic Projections in different contexts, notably at the end of No. 8, formulation of 10.4 and others and the "sorite" that follows should without a doubt come sooner in the beginning of the paragraph and possibly in the auxiliary paragraph "grassmanian". Let $C = P(F)$ be a linear subvariety of $P(E) = P$ of relative dimension $r - m - 1$ over S, i.e. of codimension $(m+1)$ in P so that F is a quotient of E locally free of rank $r - m$, $F = E/G$ where G is locally free of rank $m+1$. We have defined in the algebraic way of Chapter II a morphism

$$p_c : P - C = P(E) - P(E/G) \to P(G)$$

which we will interpret geometrically and which will be called (because of the description that follows) the *conic projection* with center C. (N.B. We assume $r - m - 1$ to be in between -1 and $r - 1$, i.e. m is between 0 and r, nothing more. For this let us begin by interpreting $P(G)$ as a closed subscheme of $\mathrm{Grass}^m(P) = \mathrm{Grass}_{r-m+1}(P)$ due to the obvious homomorphism of functors

$$P(G) \to \mathrm{Grass}_{r-m+1}(E)$$

obtained by considering for every invertible quotient G/G' of G, the locally free module of rank $(r-m)+1$ E/G' of E and the same after every base change). The above homomorphism of functors is a *monomorphism* and since the first one is proper over S, the second one separated is a closed immersion. More generally we should make explicit the closed immersions of grassmanians of G, i.e. of $P(G)$ into those of E, i.e. of $P(E)$. The image (in the sense of functors) of the obtained morphism is formed from linear subvarieties L of P, of a given dimension that *contain* C. Let us denote by $Q(C)$ this image in the case that we study (i.e. for the dimensions specified above) and identifying $P(G)$ with $Q(C)$ the morphism of conic projection

$$p_c: P - C \to Q(C) \subset \mathrm{Grass}^m(P)$$

is nothing else but the one that associates with every section of $P - C$ the unique linear subvariety L of P of codimension m containing at the same time C and the given section (of course by "containing a section" we mean that the section factors by L). If now we have $f: X \to P$, it makes sense to consider the composition

$$X - f^{-1}(C) \to P - C \to Q(C)$$

which we may call conic projection of X relative to f and with center C, denoted p_c^X or simply p_c. We shall point out that in general it is not defined over all of X, precisely it is such if and only if $f^{-1}(C) = \phi$, i.e. $f(X)$ does not meet the center C of the projection. We shall give another interpretation of this morphism in terms of constructions used in previous Nos. For this, with the notations introduced elsewhere, let us consider

$$\begin{array}{ccl} X & \xleftarrow{q} & X^{(m)}_{Q(C)} = \mathfrak{X}^{(m)} \underset{\mathrm{Grass}^m}{\mathfrak{X}} Q(C) \\ & & \downarrow p \\ & & Q(C) \end{array}$$

Let us note on the other hand that q induces an isomorphism

$$q': q^{-1}(X - f^{-1}(C)) \xrightarrow{\sim} X - f^{-1}(C)$$

and it is immediate that p_c is nothing else but $p'q'^{-1}$ where p' is the restriction of p to $q^{-1}(X - f^{-1}(C))$. We may therefore say using q' just for a simple identification that p_c is the restriction of the morhism p to $X - f^{-1}(C) \subset \mathfrak{X}^m_{Q(C)}$. For that reason it is convenient to denote again by p_c^X or p_c and to call the previous morphism [(*for instance*)] the *extended conic projection* of X relative to $f: X \to P$, with center C. In this way the properties of the restricted conic projection are reduced to those of the extended conic projection, which has supposedly been systematically studied elsewhere (cf. No. 10 and No. 12). The main question that arises is, if $S = \mathrm{Spec}(k)$ what are the properties of the conic projection of X if we take C to be generic in $\mathrm{Grass}^{m+1}(P)$ which requires that we make a base change $k \to k(\eta)$, i.e. C is then indeed a linear subvariety of $X_{k(\eta)}$; standard arguments that have already been repeated so often allow us to conclude about similar properties for the conic projections corresponding to the points of $\mathrm{Grass}^{m+1}(P)$ belonging to anon-empty open set of the said grassmanian and finally when k is infinite we conclude about the existence of a (in fact of an infinity of) C *defined over* k, i.e. a linear subvariety of P iself (without changing the base field) producing a conic projection having the said properties. We should rather group this type of general explanations with those of the same type given in No. 4, 7 and which we have already implicitly used more or less, for example in No. 13 (cf. 13.4.c) For the same reason, we should better to examine the relative properties of a sheaf F over X, taking its inverse image $F^{(m)}_{Q(C)}$ over $X^{(m)}_{Q(C)}$. Moreover, it is necessary in the precise case just described to have simpler notations, I propose $\widetilde{X}(C)$ and $\widetilde{F}(S)$ or simply $\widetilde{X}$ and $\widetilde{F}$ if there is no possibility of confusion (attention: the F here is not the same as in the beginning of this No.). Grosso modo (roughly speaking) and if we, say, assume that f is an immersion, the properties of the generic conic projection are very different, depending on whether we assume $\dim X \geq m$ or $\dim X \leq m$ even $\dim X < m$. In what follows, we consider the $C_\eta \subset P_{k(\eta)}$ corresponding to the generic point η of Grass^{m+1} and we give up making the interpretation of the obtained results in terms of "almost all the points ... "

To start with, we already have noticed in 5.3 (a 'catching up' of the general case in No. 12) that C_η cuts X_η "regularly", more precisely and more generally for every quasi-coherent F over X the section $\phi_\eta^{(m+1)}$ of the locally free module of rank $m+1$ over $X_{k(\eta)}$ whose scheme of zeros is C_η, is F-regular. By 10.2 this implies for example that the morphism $\widetilde{X}_{(C_\eta)} \to X_{k(\eta)}$ identifies $\widetilde{X}_{(C_\eta)}$ with the prescheme deduced from $X_{k(\eta)}$ by blowing up $f^{-1}(C_\eta) = X_p \times C_\eta$ in the case where $\dim f(X) \leq m$ we will also have $f^{-1}(C_\eta) = \emptyset$ and consequently $\widetilde{X}(C_\eta) \xrightarrow{\sim} X_{k(\eta)}$ is an *isomorphism* (and indeed the restricted conic projection is therefore defined over all of X a priori). Then the questions of the dimension of the fibers of $p_c\colon \widetilde{X}(C_\eta) \to Q(C_\eta)$, and the flatness of this morphism arise. We find:

Proposition 14.l. *Let us suppose that X is irreducible, more generally that for every irreducible component X_i of X the fiber of X_i at the point $f(x_i)$ (x_i = generic point of X_i) has a dimension (independent of i), which is the case for example with $d = 0$ if $f\colon X \to P$ is quasi-finite. Then*

a) *If $\dim f(X) > m$ then the dimension of the fibers of $p_c\colon \widetilde{X}(C_\eta) \to Q(C_\eta)$ are all equal to $\dim X - m$.*

b) *If $\dim f(X) \leq m$ and if the non-empty fibers of X^i over P are all of $\dim d$ then the non-empty fibers of p_c are all of dimension d so p_c is* *finite resp. quasi-finite... $f: X \to P$ is finite.*

In the case a) we know already (I hope!) that for *every* point ξ of $\mathrm{Grass}^m(P)$ the dimension of $\mathfrak{X}_\xi^{(m)}$ is at least equal to $\dim X - m$, it is in particular such if ξ comes from a point of $Q(C_\eta)$. For the opposite direction inequality note that (we place ourselves over the field $k' = k(\xi)$) since $C_{\eta k'} \subset L_\xi$ is a hyperplane of L_ξ, if the dimension of $\mathfrak{X}_\xi^{(m)} = X \times_P L_\xi$ (*was*) $\geq \dim X - m + 1$, then that of $\mathfrak{X}_\eta^{(m+1)} = X \times_P C_\eta$ would be $\geq \dim X - m$, (since the base change $k(\eta) \to k'$ transforms the last prescheme into $(X \times_P L_\xi) \times_{L_\xi} (C_{\eta k'})$). However since we have the contrary: $\dim \mathfrak{X}^{(m+1)} = \dim X - m + 1$. by No. 2 (repeated in No. 10). The case b) is treated in the same way: if we had $\dim X \times_P L \geq d+1$, or what is the same $f_{k'}(X_{k'}) \cap L$ is of dim ≥ 1 then we would have using the same argument as above that $\mathfrak{X}^{(m+1)} \neq \phi$ in contradiction with what we have remarked before 14.1.

Corollary 14.2. *Let us assume that X has dimension m and that $f: X \to P$ is finite (respectively quasi-finite), then the morphism $P_{C\eta}: X_{k(\eta)} \to Q(C_\eta)$ is finite surjective (resp. quasi-finite dominant).*

Indeed, this morphism is quasi-finite and since $\dim X_{k(\eta)} = \dim Q(C_\eta)$ is dominant if f is finite, $p_{C\eta}$ is also finite, as proper, therefore surjective, since it is dominant.

Corollary 14.3. *With the conditions of 14.1 a) if X is Cohen-Macauley then the morphism $p_C: \widetilde{X}(C_\eta) \to Q(C_\eta)$ is a Cohen-Macauley morphism and à fortiori flat.*

For the proof see the remark above on page 21 before 5. (to be corrected by Interpr.) which gives a stronger result which (to include in 14.3), taking into account that $\mathcal{C}_\xi^{(m)}$ for $\xi \in \varphi(\eta)$ are $F_{k(\eta)}$ regular.

This corollary must be modified but for simplicity we may assume that f is quasi-finite if F is a Cohen-Macauley module over X and if for every irreducible component Z of $\mathrm{Supp}\, F$ we have $\dim Z \geq m$ then $\widetilde{F}(C_\eta)$ is Cohen-Macauley (and á fortiori) flat relatively to $\varphi(C_\eta)$.

We notice that we cannot replace, to obtain the same conclusion p_C flat, the CM hypothesis on X by a simple dimension hypothesis. Let us for example assume that f is an immersion and that X is irreducible of dimension m, so that p_C is quasi-finite and since $X_{k(\eta)}$ and $Q(C_\eta)$ are irreducible of same dimensions and the second one is regular, p_C cannot be flat unless $X_{k(\eta)}$ is CM.

More delicate are the differential properties of the conic projection, notably for X smooth over k and $f: X \to P$ unramified studied in No. 12. Let us recall that outside of a subset Z of codim 1 of $Q(C)$ the morphism $p_{C\eta}$ over $\widetilde{X}(C_\eta)$ is *smooth*. And a more detailed analysis summarized in No. 12 shows (or should show if we do not do it) that if the dimensions of the components of X are $\geq m$ then outside of a subset $Z' \subset Z$ of $Q(C_\eta)$ of codimension ≥ 2, the fibers $p_C^{-1}(\xi) = X_\xi^{(m)}$ can only have at the worst ordinary singular points (in the geometric sense) and indeed (if f is an immersion and X is geometrically irreducible) at most *one* such point, the latter being necessarily rational over $k(\xi)$ – these assertions being all valid at least if k is of characteristic 0 *or* only if we replace f by$\phi_n f$ $(n \geq 2)$ as in No. 9.

It is also appropriate to give the differential properties of $P_{C\eta}$ in the case where dim $X \leq m$ and consequently where $P_{C\eta}$ is defined over $X_{k(\eta)}$. I restrict myself to indicating the following properties. The proof should be easy and is left to Dieudonné (or Blass). [Interpr.]

Proposition 14.4. *Let us suppose that $f\colon X \to P$ is unramified and that* dim $X \leq m$*. Let T be a finite subscheme of X. Then*

a) If f is an immersion, the restriction of p_C to $T_{k(\eta)}$ is radical, i.e. "geometrically injective". If in addition Y a closed subset of X of dimension $\leq (m-1)$ we have

$$p_{C\eta}^{-1}(p_{C\eta}(Y_{k(\eta)})) \cap T_{k(\eta)} = \phi = \text{ empty set}$$

b) If X is smooth at the points of T then $p_{C\eta}$ is unramified at all the points $T_{k(\eta)}$ and also at the points of

$$p_{C\eta}^{-1} p_{C\eta}(T_{k(\eta)})$$

Proposition 14.5. *Let us suppose that* dim $X \leq m-1$, $f\colon X \to P$ *an immersion, finally X separable over k. Let Y_η be the scheme image of $X_{k(\eta)}$ in $Q(C_\eta)$. Then the induced morphism $p_{C\eta}\colon X_{k(\eta)} \to Y_\eta$ is birational and for every point x of $X_{k(\eta)}$ over a closed point of X, $p_{c\eta}$ is étale at x and even at the points of $p'_{C_\eta} p'_{C_\eta}(x)$.*

Let us note the following consequence:

Corollary 14.6. *Let X be an algebraic projective scheme irreducible and separable of dimension n over an infinite field k. Then there exists a birational isomorphism of X onto a hypersurface of P^{n+1}.*

We will avoid believing that, even if X is a closed smooth geometrically irreducible subset of P of dimension $m-1=n$, the conic projection (P_{c_η}) is necessarily an immersion. Indeed if k is infinite this would imply that there exists a C rational over k having the same property, then that X is isomorphic to a non-singular hypersurface of P^{n+1}. But even for $n=1$ (thus X is an algebraic projective curve smooth and connected over an algebraically closed field) it is easy to construct examples when X cannot be in a P^2. Also in 14.4 (in the same way) we will avoid to confuse the given statement with the assertion (in general false) that p_c is itself a monomorphism (previous counterexample or even more obvious counterexample if X is smooth of dimension m), or that P_c should be unramified. For the last point to convince ourselves let us take X to be a closed smooth subscheme irreducible and of dimension m (over k algebraically closed say such that if we had $X \to Q \cong p^m$ unramified, it would be étale because of dimensions, but we can prove (see Ch. VIII) that this implies that $X \xrightarrow{\sim} P^m$ (P^m being "simply connected"). The intuitive geometric meaning of 14.4 is that the ramification set of p_{C_η} is "variable" over k more precisely the ramification set of p_{C_ξ} for a variable ξ in an open set of $\mathrm{Grass}^{m+1}(\bar{k})$ varies in $X(\bar{k})$ and does not admit any "fixed point"... Of course, in order to justify in the present No. the passage from a generic point η to a neighborhood of $\mathrm{Grass}^{m+1}(P)$ and also, to be able if needed to take back our general considerations of 7.1, we have to consider the diagrams:

$$\begin{array}{ccc} X & \longleftarrow & \widetilde{X}(C) \\ \big\downarrow & & \big\downarrow \\ X & \longleftarrow & Q(C) \end{array}$$

obtained (with the help) using different $C \in \mathrm{Grass}^{m+1}(S)$ and more generally those obtained after a base change $T \to S$ for points $\xi \in \mathrm{Grass}^{m+1}(T)$

$$\begin{array}{ccc} X_T & \longleftarrow & \widetilde{X}(C_\xi) = \widetilde{X}_T(C_\xi) \\ \big\downarrow & & \big\downarrow \\ T & \longleftarrow & Q(C_\xi) \end{array}$$

as deduced by base change $\xi: T \to \mathrm{Grass}^{m+1} J(P) = T$, of the *universal* diagram (relative to the canonical point of Grass^{m+1} in T):

$$\begin{array}{ccc} X_{\mathcal{T}} & \longleftarrow & \widetilde{X}(\mathcal{C}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{T} & \longleftarrow & Q(\mathcal{C}) \end{array}$$

where $\mathcal{C}$ is the canonical linear subvariety of $P_{\mathcal{T}}$. Then the above $\widetilde{X}(C_\eta) \to Q(C_\eta)$ is nothing else but the morphism of generic fibers for the $\mathcal{T}$ morphism $\widetilde{X}(\mathcal{C}) \to Q(\mathcal{C})$ of this last diagram and every constructible property for the morphism of generic fibers implies the same property for neighboring fibers. From the notational point of view, Q should be looked at (and even introduced) as the name of the natural morphism of functors $\mathrm{Grass}^{m+1}(P) \to$ Sub-preschemes of $\mathrm{Grass}^m(P)$.