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Quaternion Recurrence Relations

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1. Introduction

In a hopeful moment it was suggested [4] that the investigtion of complex Fibonacci numbers undertaken in [3] might be extended to quaternions (hypercomplex numbers). It is the purpose of this paper to examine the degree to which this expectation is realized. Harman's ideas in [3] are extended in another way in [7].

A *quaternion* Q with real components a_0, a_1, a_2, a_3 and basis 1, i, j, k is a number of the form

$$Q = 1a_0 + ia_1 + ja_2 + ka_3 \qquad (1a_0 = a_0) \tag{1.1}$$

where

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k = -ji, jk = i = -kj, ki = j = -ik. \end{cases} \tag{1.2}$$

The *conjugate* $\bar{Q}$ of Q is

$$\bar{Q} = 1a_0 - ia_1 - ja_2 - ka_3 \tag{1.3}$$

and the *norm* is

$$Q\bar{Q} = a_0^2 + a_1^2 + a_2^2 + a_3^2. \tag{1.4}$$

Quaternions form an associative, non-commutative division algebra of rank 4 over the reals [1]. The quarternion algebra has no divisors of zero. Complex numbers form a sub-algebra of quaternions when $a_2 = a_3 = 0$.

Next, define the sequence of numbers $\{u_n\}$ by the recurrence relation

$$u_{n+2} = p\,u_{n+1} - q\,u_n \qquad (n \geq 0) \tag{1.5}$$

$$u_0 = 0,\ u_1 = 1 \tag{1.6}$$

where p and q are arbitrary non-zero real numbers.

> When $p = 1$, $q = -1$, $u_n = F_n$ (the n-th *Fibonacci number*). If $p = 2$, $q = -1$, then $u_n = P_n$ (the n-th *Pell number*).

Other examples of (1.5), with (1.6), do not especially concern us here.

Particular usage of (1.5) and (1.6), which will be important in subsequent work on generalization, is made for the sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$ and $\{T_n\}$ defined by the following recurrence relations and initial conditions:

$$U_{h+2} = p_1 U_{h+1} - q_1 U_h \qquad U_0 = 0,\ U_1 = 1 \tag{1.7}$$

$$V_{\ell+2} = p_2 V_{\ell+1} - q_2 V_\ell \qquad V_0 = 0,\ V_1 = 1 \tag{1.8}$$

$$W_{m+2} = p_3 W_{m+1} - q_3 W_m \qquad W_0 = 0,\ W_1 = 1 \tag{1.9}$$

$$T_{n+2} = p_4 T_{n+1} - q_4 T_n \qquad T_0 = 0,\ T_1 = 1. \tag{1.10}$$

Though in our defining recurrence relation (1.5) we restrict n to be ≥ 0, the use of negative subscripts may be occasionally useful. For instance, one can easily establish that $F_{-n} = (-1)^{n-1} F_n$ so that $F_{-1} = 1$.

2. Quaternions, Fibonacci Numbers and the Symbol G

Extending the ideas if [3], let us define the recurrence relations

$$\begin{cases} G(h+2,\ell,m,n) = G(h+1,\ell,m,n) + G(h,\ell,m,n) \\ G(h,\ell+2,m,n) = G(h,\ell+1,m,n) + G(h,\ell,m,n) \\ G(h,\ell,m+2,n) = G(h,\ell,m+1,n) + G(h,\ell,m,n) \\ G(h,\ell,m,n+2) = G(h,\ell,m,n+1) + G(h,\ell,m,n) \end{cases} \quad h,\ell,m,n \geq 0 \tag{2.1}$$

with initial conditions

$$\begin{cases} G(0,0,0,0) = 0,\ G(1,0,0,0) = 1,\ G(0,1,0,0) = i, \\ G(0,0,1,0) = j,\ G(0,0,0,1) = k, \\ G(1,1,0,0) = 1+i, \ldots, G(0,0,1,1) = j+k \\ G(1,1,1,0) = 1+i+j, \ldots, G(0,1,1,1) = i+j+k \\ G(1,1,1,1) = 1+i+j+k. \end{cases} \tag{2.2}$$

Inductive, or other, proofs may be employed to derive the following results for the symbol G:

$$G(h,0,0,0)=F_h,\ G(0,\ell,0,0)=iF_\ell,\ G(0,0,m,0)=jF_m,$$
$$G(0,0,0,n)=kF_n \tag{2.3}$$
$$G(h,1,0,0)=F_h+iF_{h+1} \tag{2.4}$$
$$G(h,\ell,0,0)=F_hF_{\ell+1}+iF_{h+1}F_\ell \tag{2.5}$$
$$G(h,1,1,0)=F_h+(i+j)F_{h+1} \tag{2.6}$$
$$G(h,,\ell,1,0)=F_hF_{\ell+1}+iF_{h+1}F_\ell+jF_{h+1}F_{\ell+1} \tag{2.7}$$
$$G(h,\ell,m,0)=F_hF_{\ell+1}F_{m+1}+iF_{h+1}F_\ell F_{m+1}$$
$$+jF_{h+1}F_{\ell+1}F_m \tag{2.8}$$
$$G(h,1,1,1)=F_h+(i+j+k)F_{h+1} \tag{2.9}$$
$$G(h,\ell,1,1)=F_hF_{\ell+1}+iF_{h+1}F_\ell+(j+k)F_{h+1}F_{\ell+1} \tag{2.10}$$
$$G(h,\ell,m,1)=F_hF_{\ell+1}F_{m+1}+iF_{h+1}F_\ell F_{m+1}$$
$$+jF_{h+1}F_{\ell+1}F_m+kF_{h+1}F_{\ell+1}F_{m+1} \tag{2.11}$$
$$G(h,\ell,m,n)=F_hF_{\ell+1}F_{m+1}F_{n+1}+iF_{h+1}F_\ell F_{m+1}F_{n+1}$$
$$+jF_{h+1}F_{\ell+1}F_mF_{n+1}+kF_{h+1}F_{\ell+1}F_{m+1}F_n. \tag{2.12}$$

Thus, $G(h,\ell,m,n)$ is a quaternion with quadruple-product components

$$F_hF_{\ell+1}F_{m+1}F_{n+1},\quad F_{h+1}F_\ell F_{m+1}F_{n+1},$$
$$F_{h+1}F_{\ell+1}F_mF_{n+1},\quad F_{h+1}F_{\ell+1}F_{m+1}F_n.$$

(Compare this detail with corresponding information given in [4].)

Clearly, all the expressions for the G's are quaternions whose components are Fibonacci numbers ($F_0=0,\ F_1=1$) or products of Fibonacci numbers. Obviously there is a pattern in the composition of these components.

Some numerical examples of (2.12) are the quaternions

$$G(2,0,1,3)=3+6j+4k,\qquad G(1,2,3,4)=30+15i+20j+18k,$$
$$G(2,3,1,4)=15+20i+30j+18k.$$

If one imagines m and n in (2.1) to be non-existent, then our results reduce to those given in [3] for complex Fibonacci numbers. Indeed our results (2.3), (2.4), (2.5) are identical with the nice results (2.5), (2.6), (2.7) respectively in [3], *mutatis mutandis*.

Perhaps we shuld offer a proof of one of the above expressions relating to G. Choose (2.12).

Proof of (2.12): Proceed by induction on n in the first instance. Assume

$$G(h,\ell,m,n)=F_nG(h,\ell,m,1)+F_{n-1}G(h,\ell,m,0). \tag{α}$$

This is clearly true when $n = 0, n = 1$ (since $F_0 = 0$, $F_{-1} = 1$).

For $n = k$ (constant), (α) is

$$G(h, \ell, m, k) = F_k G(h, \ell, m, 1) + F_{k-1} G(h, \ell, m, 0). \qquad (\beta)$$

Then

$$\begin{aligned} G(h,\ell, m, k+1) &= G(h, \ell, m, k) + G(h, \ell, m, k-1) \qquad \text{by (2.1)} \\ &= F_k G(h, \ell, m, 1) + F_{k-1} G(h, \ell, m, 0) \\ &+ F_{k-1} G(h, \ell, m, 1) + F_{k-2}(h, \ell, m, 0) \qquad \text{by } (\beta) \\ &= (F_k + F_{k-1}) G(h, \ell, m, 1) + (F_{k-1} + F_{k-2}) G(h, \ell, m, 0) \\ &= F_{k+1} G(h, \ell, m, 1) + F_k G(h, \ell, m, 0) \quad \text{since } F_{k+1} = F_k + F_{k-1}. \end{aligned}$$

Consequently, (α) is true for $n = k+1$ and hence for all n.

Next

$$\begin{aligned} G(h, \ell, m, n) &= F_n\{F_h F_{\ell+1} F_{m+1} + iF_{h+1} F_\ell F_{m+1} + jF_{h+1} F_{\ell+1} F_m \\ &\qquad + kF_{h+1} F_{\ell+1} F_{m+1}\} \\ &\quad + F_{n-1}\{F_h F_{\ell+1} F_{m+1} + iF_{h+1} F_\ell F_{m+1} + jF_{h+1} F_{\ell+1} F_m\} \\ &\qquad \text{by } (\alpha), \text{(2.8) and (2.11)} \\ &= F_h F_{\ell+1} F_{m+1} F_{n+1} + iF_{h+1} F_\ell F_{m+1} F_{n+1} + jF_{h+1} F_{\ell+1} F_m F_{n+1} \\ &\qquad + kF_{h+1} F_{\ell+1} F_{m+1} F_n \end{aligned}$$

on using $F_n + F_{n-1} = F_{n+1}$.

So far so good. When, however, we seek to obtain a recurrence relation for the quaternion symbol $G(h, \ell, m, n)$ the algebra becomes somewhat complicated. After expansion using (2.12), and appropriate algebraic manipulation and grouping, we find

$$\begin{aligned} &G(h+1, \ell+1, m+1, n+1) \qquad (2.13) \\ &= F_{h+1} F_{\ell+2} F_{m+2} F_{n+2} + iF_{h+2} F_{\ell+1} F_{m+2} F_{n+2} + jF_{h+2} F_{\ell+2} F_{m+1} F_{n+2} \\ &\quad + kF_{h+2} F_{\ell+2} F_{m+2} F_{n+1} \\ &= F_{h+1}(F_{\ell+1} + F_\ell)(F_{m+1} + F_m)(F_{n+1} + F_n) + iF_{\ell+1}(..)(..)(..) \\ &\quad + jF_{m+1}(..)(..)(..) + kF_{n+1}(..)(..)(..) \\ &= (1 + i + j + k) F_{h+1} F_{\ell+1} F_{m+1} F_{n+1} + G(\ell, h, n, m) + G(m, n, h, \ell) \\ &\qquad + G(n, m, \ell, h) \\ &\quad + (1+i) F_{h+1} F_{\ell+1} F_m F_n + (1+j) F_{h+1} F_\ell F_{m+1} F_n \\ &\qquad + (1+k) F_{h+1} F_\ell F_m F_{n+1} \\ &\quad + (i+j) F_h F_{\ell+1} F_{m+1} F_n + (j+k) F_h F_\ell F_{m+1} F_{n+1} \\ &\qquad + (i+k) F_h F_{\ell+1} F_m F_{n+1} \\ &\quad + F_{h+1} F_\ell F_m F_h + iF_h F_{\ell+1} F_m F_n + jF_h F_\ell F_{m+1} F_n + kF_h F_\ell F_m F_{n+1} \end{aligned}$$

$$
\begin{aligned}
&= (1+i+j+k)F_{h+1}F_{\ell+1}F_{m+1}F_{n+1} + (i+j+k)F_hF_{\ell+1}F_{m+1}F_{n+1}+ \\
&\quad + (1+j+k)F_{h+1}F_\ell F_{m+1}F_{n+1} \\
&\quad + (1+i+k)F_{h+1}F_{\ell+1}F_mF_{n+1} + (1+i+j)F_{h+1}F_{\ell+1}F_{m+1}F_n \\
&\quad + (1+i)F_{h+1}F_{\ell+1}F_mF_n + (1+j)F_{h+1}F_\ell F_{m+1}F_n \\
&\qquad\quad + (1+k)F_{h+1}F_\ell F_mF_{n+1} \\
&\quad + (i+j)F_hF_{\ell+1}F_{m+1}F_n + (i+k)F_hF_{\ell+1}F_mF_{n+1} \\
&\qquad\quad + (j+k)F_hF_\ell F_{m+1}F_{n+1} \\
&\quad + F_{h+1}F_\ell F_mF_n + iF_hF_{\ell+1}F_mF_n + jF_hF_\ell F_{m+1}F_n + kF_hF_\ell F_mF_{n+1} \\
&= \Sigma G(a,b,c,d)F_{h+a}F_{\ell+b}F_{m+c}F_{n+d}
\end{aligned}
$$

where each of a, b, c, d equals 0 or 1, and where the initial conditions (2.2) have been applied. Included additionally in the nice summation expression in the final line is the coefficient $G(0,0,0,0) = 0$.

Repetition of (2.13) with (2.12) yields

$$
\begin{aligned}
&G(h+2,\ell+2,m+2,n+2) \qquad\qquad (2.14) \\
&= (1+i+j+k)F_{h+2}F_{\ell+2}F_{m+2}F_{n+2} + G(\ell+1,h+1,n+1,m+1) \\
&\quad + G(m+1,n+1,h+1,\ell+1) + G(n+1,m+1,\ell+1,h+1) \\
&\quad + (1+i)F_{h+2}F_{\ell+2}F_{m+1}F_{n+1} + (1+j)F_{h+2}F_{\ell+1}F_{m+2}F_{n+1} \\
&\qquad\quad + (1+k)F_{h+2}F_{\ell+1}F_{m+1}F_{n+2} \\
&\quad + (i+j)F_{h+1}F_{\ell+2}F_{m+2}F_{n+1} + (j+k)F_{h+1}F_{\ell+1}F_{m+2}F_{n+2} \\
&\qquad\quad + (i+k)F_{h+1}F_{\ell+2}F_{m+1}F_{n+2} \\
&\quad + F_{h+2}F_{\ell+1}F_{m+1}F_{n+1} + iF_{h+1}F_{\ell+2}F_{m+1}F_{n+1} \\
&\qquad\quad + jF_{h+1}F_{\ell+1}F_{m+2}F_{n+1} + kF_{h+1}F_{\ell+1}F_{m+1}F_{n+2} \\
&= (1+i+j+k)\{F_{h+2}F_{\ell+2}F_{m+2}F_{n+2} + 7F_{h+1}F_{\ell+1}F_{m+1}F_{n+1} \\
&\qquad\quad + F_hF_{\ell+1}F_{m+1}F_{n+1} \\
&\quad + F_{h+1}F_\ell F_{m+1}F_{n+1} + F_{h+1}F_{\ell+1}F_mF_{n+1} + F_{h+1}F_{\ell+1}F_{m+1}F_n\} \\
&\quad + 6G(h,\ell,m,n) + 2\{G(\ell,h,n,m) + G(m,n,h,\ell) + G(n,m,\ell,h)\} \\
&\quad + (1+i)\{F_{h+1}F_{\ell+1}F_mF_n + 3F_hF_\ell F_{m+1}F_{n+1}\} + (1+j)\cdot \\
&\qquad\quad \{F_{h+1}F_\ell F_{m+1}F_n + 3F_hF_{\ell+1}F_mF_{n+1}\} \\
&\quad + (1+k)\{F_{h+1}F_\ell F_mF_{n+1} + 3F_hF_{\ell+1}F_{m+1}F_n\} + (i+j)\cdot \\
&\qquad\quad \{F_hF_{\ell+1}F_{m+1}F_n + 3F_{h+1}F_\ell F_mF_{n+1}\} \\
&\quad + (j+k)\{F_hF_\ell F_{m+1}F_{n+1} + 3F_{h+1}F_{\ell+1}F_mF_n\} + (i+k)\cdot \\
&\qquad\quad \{F_hF_{\ell+1}F_mF_{n+1} + 3F_{h+1}F_\ell F_{m+1}F_n\} \\
&\quad + (i+j+k)F_{h+1}F_\ell F_mF_n + (1+j+k)F_hF_{\ell+1}F_mF_n \\
&\qquad\quad + (1+i+k)F_hF_\ell F_{m+1}F_n \\
&\quad + (1+i+j)F_hF_\ell F_mF_{n+1}
\end{aligned}
$$

eventually.

Particulr illustrations of this recurrence relation, which may be verified using (2.12) and (2.13), are

$$G(3,2,2,3) = 24 + 18i + 18j + 24k$$

and

$$G(4,3,2,5) = 144 + 160i + 120j + 150k.$$

Equation (2.13), and more especially (2.14), are only established after painstaking and arduous effort. Other forms of the expressions exist, but the ones given seem to be instructive, if somewhat inelegant, options. Various permutations of h,ℓ, m,n in the symbols $G(.,.,.,.)$ may for instance be employed in an alternative representation. Scrutinizing the correspondence between the occurrence of the quaternionic units 1, i, j, k and the subscripts h, ℓ, m, n in that order reveals an underlying unifying pattern, as we have come to expect.

Variation of the form in (2.14) may be derived as a special case of (3.14).

If we imagine the numbers m, n and the quaternion units j, k to be entirely absent from our considerations, we are then dealing with complex numbers and results (2.13) and (2.14) lead to those simpler recurrences given in [3].

3. Generalization

Let us now turn our attention to a generalization of the results in the preceding section.

To do this we are guided by the model for generalizing Gaussian Fibonacci numbers, i.e. complex numbers with Fibonacci components, employed in [7]. (This paper, [7], establishes summation identities involving the products of combinations of Fibonacci numbers and polynomials, Pell numbers and polynomials, Chebyshev polynomials, and sine functions.)

Define

$$\begin{cases} g(h+2,\ell,m,n) = p_1 g(h+1,\ell,m,n) - q_1 g(h,\ell,m,n) \\ g(h,\ell+2,m,n) = p_2 g(h,\ell+1,m,n) - q_2 g(h,\ell,m,n) \\ g(h,\ell,m+2,n) = p_3 g(h,\ell,m+1,n) - q_3 g(h,\ell,m,n) \\ g(h,\ell,m,n+2) = p_4 g(h,\ell,m,n+1) - q_4 g(h,\ell,m,n) \end{cases} \quad h,\ell,m,n \geq 0 \tag{3.1}$$

with

$$\begin{cases} g(0,0,0,0) = 0, g(1,0,0,0) = 1, g(0,1,0,0,0) = i, g(0,0,1,0) = j, \\ g(0,0,0,1) = k \\ g(1,1,0,0) = p_2 + ip_1, g(1,0,1,0) = p_3 + jp_1, g(0,1,1,0) = ip_3 + jp_2, \ldots \\ g(1,1,1,0) = p_2p_3 + ip_1p_3 + jp_1p_2, g(1,0,1,1) = p_3p_4 + jp_1p_4 + kp_1p_3, \ldots \\ g(1,1,1,1) = p_2p_3p_4 + ip_1p_3p_4 + jp_1p_2p_4 + kp_1p_2p_3. \end{cases} \tag{3.2}$$

The pattern of the existence of subscripts of the real numbers p, in relation to the presence of the quaternion units 1, i, j, k, and to the presence of the 1's and 0's in the $g(.,.,.,.)$, should be clear from the information given in (3.2).

Analogously to the results in the previous section we derive

$$g(h,0,0,0) = U_h,\ g(0,\ell,0,0) = iV_\ell,$$
$$g(0,0,m,0) = jW_m, g(0,0,0,n) = kT_n \tag{3.3}$$
$$g(h,1,0,0) = U_hV_2 + iU_{h+1} \tag{3.4}$$
$$g(h,\ell,0,0) = U_hV_{\ell+1} + iU_{h+1}V_\ell \tag{3.5}$$
$$g(h,1,1,0) = U_hV_2W_2 + iU_{h+1}W_2 + jU_{h+1}V_2 \tag{3.6}$$
$$g(h,\ell,1,0) = U_hV_{\ell+1}W_2 + iU_{h+1}V_\ell W_2 + jU_{h+1}V_{\ell+1} \tag{3.7}$$
$$\begin{aligned} g(h,\ell,m,0) &= U_hV_{\ell+1}W_{m+1} + iU_{h+1}V_\ell W_{m+1} \\ &\quad + jU_{h+1}V_{\ell+1}W_m \end{aligned} \tag{3.8}$$
$$\begin{aligned} g(h,1,1,1) &= U_hV_2W_2T_2 + iU_{h+1}W_2T_2 + jU_{h+1}V_2T_2 \\ &\quad + kU_{h+1}V_2W_2 \end{aligned} \tag{3.9}$$
$$\begin{aligned} g(h,\ell,1,1) &= U_hV_{\ell+1}W_2T_2 + iU_{h+1}V_\ell W_2T_2 + jU_{h+1}V_{\ell+1}T_2 \\ &\quad + kU_{h+1}V_{\ell+1}W_2 \end{aligned} \tag{3.10}$$
$$\begin{aligned} g(h,\ell,m,1) &= U_hV_{\ell+1}W_{m+1}T_2 + iU_{h+1}V_\ell W_{m+1}T_2 + jU_{h+1}V_{\ell+1}W_mT_2 \\ &\quad + kU_{h+1}V_{\ell+1}W_{m+1} \end{aligned} \tag{3.11}$$
$$\begin{aligned} g(h,\ell,m,n) &= U_hV_{\ell+1}W_{m+1}T_{n+1} + iU_{h+1}V_\ell W_{m+1}T_{n+1} \\ &\quad + jU_{h+1}V_{\ell+1}W_mT_{n+1} + kU_{h+1}V_{\ell+1}W_{m+1}T_n. \end{aligned} \tag{3.12}$$

Proofs of these identities are readily forthcoming.

Paralleling the reasoning used in (2.13), and with (3.2), we determine its analogue to be the quaternion

$$\begin{aligned} g(h+1,\ell+1,m+1,n+1) &= g(1,1,1,1)U_{h+1}V_{\ell+1}W_{m+1}T_{n+1} \\ &\quad - q_1g(0,1,1,1)U_hV_{\ell+1}W_{m+1}T_{n+1} \qquad (3.13) \\ &\quad - q_2g(1,0,1,1)U_{h+1}V_\ell W_{m+1}T_{n+1} - q_3g(1,1,0,1)U_{h+1}V_{\ell+1}W_mT_{n+1} \\ &\quad - q_4g(1,1,1,0)U_{h+1}V_{\ell+1}W_{m+1}T_n \\ &\quad + q_1q_2g(0,0,1,1)U_hV_\ell W_{m+1}T_{n+1} + q_1q_3g(0,1,0,1)U_hV_{\ell+1}W_mT_{n+1} \\ &\quad + q_1q_4g(0,1,1,0)U_hV_{\ell+1}W_{m+1}T_n + q_2q_3g(1,0,0,1)U_{h+1}V_\ell W_mT_{n+1} \\ &\quad + q_2q_4g(1,0,1,0)U_{h+1}V_\ell W_{m+1}T_n + q_3q_4g(1,1,0,0)U_{h+1}V_{\ell+1}W_mT_n \\ &\quad - q_2q_3q_4U_{h+1}V_\ell W_mT_n - iq_1q_3q_4U_hV_{\ell+1}W_mT_n \\ &\quad - jq_1q_2q_4U_hV_\ell W_{m+1}T_n - kq_1q_2q_3U_hV_\ell W_mT_{n+1} \\ &= \Sigma(-1)^{a+b+c+d}q_1^{1-a}q_2^{1-b}q_3^{1-c}q_4^{1-d}g(a,b,c,d)U_{h+a}V_{\ell+b}W_{m+c}T_{n+d} \end{aligned}$$

where $a,b,c,d = 0$ or 1. Putting $p_i = 1$, $q_i = -1$ $(i = 1,2,3,4)$ in (3.13) (so that $U_n = V_n = W_n = T_n = F_n$) brings us back to the case for Fibonacci numbers set out in (2.13). Substitution of $p_i = 2$, $q_i = -1$ in (3.13) (so that $U_n = V_n = W_n = T_n = P_n$) produces the corresponding situation for Pell numbers.

Numerical checking should, from time to time, be carried out. For example, if $h = \ell = m = n = 2$ in (3.12), we have on simplification

$$\begin{aligned} g(2,2,2,2) &= p_1p_2p_3p_4g(1,1,1,1) - p_1p_2p_3q_4g(1,1,1,0) - (3 \text{ similar terms}) \\ &\quad + p_1p_2q_3q_4g(1,1,0,0) + (5 \text{ similar terms}) \\ &\quad - p_1q_2q_3q_4g(1,0,0,0) - (3 \text{ similar terms}) \\ &= \Sigma(-1)^{a+b+c+d} q_1^{1-a} q_2^{1-b} q_3^{1-c} q_4^{1-d} g(a,b,c,d) U_{1+a} V_{1+b} W_{1+c} T_{1+d} \\ &\qquad \text{as in } (3.13) \\ &= 8(1+i+j+k) \quad \text{when } p_i = 1,\ q_i = -1\,(i = 1,2,3,4) \\ &= G(2,2,2,2) \quad \text{by } (2.12) \end{aligned}$$

which we may later verify by using (3.14) with (3.15) whebn $h = \ell = m = n = 0$.

Coming next to the expression for $g(h+2, \ell+2, m+2, n+2)$ we naturally get involved in some demanding algebra. To conserve space, the computational details, which can be supplied on request, are herewith suppressed. Eventually, with the aid of (3.1) and (3.2), we obtain the 81-term expression

$$\begin{aligned} g(h+2, \ell+2, m+2, n+2) & \\ &= \Sigma(-1)^{u+v+w+t} q_1^{1-u} q_2^{1-v} q_3^{1-w} q_4^{1-t} g(u,v,w,t) \cdot \\ & \Sigma(-1)^{a+b+c+d} p_1^a p_2^b p_3^c p_4^d q_1^{a'} q_2^{b'} q_3^{c'} q_4^{d'} U_{h+a''} V_{\ell+b''} W_{m+c''} T_{n+d''} \end{aligned} \tag{3.14}$$

in which

$$\left\{ \begin{array}{l} u = 1 \Longrightarrow \left\{ \begin{array}{llll} a' = 1-a & b' = 1-b & c' = 1-c & d' = 1-d \\ a'' = a & b'' = b & c'' = c & d'' = d \end{array} \right. \\ \\ u = 0 \Longrightarrow \left\{ \begin{array}{llll} a = a' = 0 & b' = 1-b & c' = 1-c & d' = 1-d \\ a'' = 1 & b'' = b & c'' = c & d'' = d \end{array} \right. \end{array} \right. \tag{3.15}$$

with similar restrictions on v, w, t $(a, b, c, d, a', b', c', d', a'', b'', c'', d'' = 0$ or $1)$. Included in the summation is the factor $q_1q_2q_3q_4\, g(0,0,0,0) = 0$, by (3.2), which is absent from the expansion.

For example, the term involving $g(1,0,1,0)$ in (3.14) is

$$q_2q_4g(1,0,1,0) \left\{ \begin{array}{ll} p_1p_3U_{h+1}V_{\ell+1}W_{m+1}T_{n+1} & -p_1q_3U_{h+1}V_{\ell+1}W_mT_{n+1} \\ -p_3q_1U_hV_{\ell+1}W_{m+1}T_{n+1} & +q_1q_3U_hV_{\ell+1}W_mT_{n+1} \end{array} \right\}.$$

If we set $p_i = 1$, $q_i = -1$ $(i = 1,2,3,4)$ in (3.14) whence $U_n = V_n = W_n = T_n = F_n$, we deduce an alternative summation form for Fibonacci numbers in (2.14). Similarly, when $p_i = 2$, $q_i = -1$ are substituted in (3.14), the corresponding expression for Pell numbers is derived, since $U_n = V_n = W_n = T_n = P_n$ then.

Nowhere in our calculations (except in (1.4)) have we had recourse to the specific multiplicative features of the quaternion units given in (1.2). But, if we desired to try to emulate the applications in [3], to obtain relationships among the Fibonacci numbers we should probably have need to use expressions like

$$(1 + i + j + k)^{-1} = \frac{1}{4}(1 - i - j - k)$$

(cf. (1.3) and (1.4)) which involves the employment of (1.2). Frailty of the flesh causes us to baulk at the daunting prospect of this development of [3], if indeed it were possible.

Note on Nomenclature: The quaternions with Fibonacci components which we have been discussing might generically be referred to as *Fibonacci quaternions and generalized Fibonacci quaternions*, though the author has elsewhere ([6], reference 2) defined these concepts somewhat differently. Likewise we might speak loosely of *Pell quaternions and generalized Pell quaternions.*

4. Concluding Comments

1. Quaternions and Geometry

Comparison might be made between the four components of $G(h, \ell, m, n)$ occurring in (2.12) and the coordinates of a point in Euclidean 4-space (i.e., space of 4 dimensions) given in [4].

Quaternions can be used in the treatment of a projective model of elliptic (non-Euclidean) 3-space. See [1]. Projective geometry over the quaternions (*quaternion geometry*), where each of the coordintes is a quaternion, is also a subject for study.

For other information on quaternions relevant to this paper consult [5] and [6], which also give reference to earlier articles on quaternions in that journal.

2. Octaves

No musical connotation is signified by the use of "octaves" in this context!

An *octave* (or *Cayley number*) is derived from quaternions in the following way. Firstly, we introduce a new symbol e ($e^2 = -1$). Octaves are then the numbers

$$P + Qe \tag{4.1}$$

where P and Q are quaternions subject to the conventions

$$\begin{cases} P + Oe = P \\ O + Qe = Qe. \end{cases} \tag{4.2}$$

The basis of the octave algebra is 1, i, j, k, e, ie, je, ke. The multiplication table of the octave basis elements is given in [1]. Like quaternions, octves have a norm (i.e., they form a normed division algebra over the reals of rank 8). Octave algebra is non-commuttive and non-associative. (Like quaternions, octaves form a vector space.)

Just as quaternions have geometric applications, so do octaves, giving rise to *octave geometry*.

Obviously, quaternions are a sub-algebra of the system of octaves. On the evidence of the work in this article, in which results for complex numbers were extended to corresponding results for quaternions, it would seem that with patience and adequate motivation our results for quaternions could be generalized to results about octaves.

Further information on octaves may be found in Artzy [1], Cayley [2], van der Blij [8], and Yaglom [9].

3. Historical Note

Quaternions, as is well-known, were the discovery of the Irish mathematician and astronomer, Hamilton, though this significant breakthrough in aglebra did not come to him without much mental travail. One might consult "The Mathematical Papers of Sir William Rowan Hamilton", Vol. III Algebra, edited by H. Halberstam and R. E. Ingram (C.U.P. 1967) for the details. [This volume also contains some useful historical information on octaves.]

However, Gauss had earlier, in 1819 or 1820, set down the multiplication table for quaternions, but he did not publish it. Nor did he develop the algebra of quaternions.

Hamilton, on pp. xv and xvi of the source quoted above, describes his electrifying discovery of quaternion multiplication in a letter to his son, Archibald. While walking along the Royal Canal, Dublin, with his wife on Monday, 16th October, 1843, on his way to preside at a Council meeting of the Royal Irish Academy, illumination suddenly came upon him. He records:

> "An electric circuit seemed to close; and a spark flashed forth,... I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k;
>
> $$i^2 = j^2 = k^2 = ijk = -1, \ldots$$ "

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