notation established above the conditions on $\vec{u_0}$ and $\vec{u_1}$ that we add are that the functions h_0 and h_1 should also satisfy

$$a_{0i}b_{1i} - a_{1i}b_{0i} > 0 \text{ , and} \tag{60}$$

$$a_{0i} > 0 \text{ for each } i = 6, 1, 2 \text{ .} \tag{61}$$

Using the computer algebra system Maple, we can show that there are at least 4 potential functions $\phi_1, \ldots, \phi_4$ satisfying (60) and (61). Moreover, the divergences $\vec{\nabla} \cdot \phi_1, \ldots, \vec{\nabla} \cdot \phi_4$ are linearly independent Weil divisors and span a subgroup of $\operatorname{Pic} X$ of rank 3. Any convex combination of $\phi_1, \ldots, \phi_4$ also satisfies (60) and (61). So there is a 3-dimensional convex subset of $\operatorname{Pic} X \otimes \mathbb{Q}$ which parametrizes the determinants of a class of possible risk functions. This set includes V and SV . There is a convex subset of dimension 4 of the group of $\mathbb{Q}$-Weil divisors with the same property.

5. Conclusion

We have introduced a toric variety associated to a portfolio and have shown that the traditional risk functions of Markowitz can be mapped to direct sums of line bundles on this variety. The variance function maps to a free vector bundle, the semivariance to a direct sum of possibly independent line bundles. It is hoped that other useful risk functions can be discovered by these means. The Picard group parametrizes the line bundles up to isomorphism and on a toric variety the Picard group is finitely generated. Therefore the set of all vector bundles that are direct sums of line bundles is a free abelian group of finite rank. Perhaps it is possible to use this set to parametrize a family of new risk functions.

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Equation (54) says that $b_{0i} = a_{1i}$. Set

$$B_i = \begin{bmatrix} \eta_i \\ \eta_{i-1} \end{bmatrix} \text{ and } B_i^{-1} = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix} .$$

Then

$$b_{0i} = (\gamma_i, \delta_i) = \begin{bmatrix} u_{0i} \\ u_{0,i-1} \end{bmatrix} .$$

Writing down all 6 equations, we have the matrix equation

$$\begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & \delta_1 \\ \delta_2 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & \delta_3 & \gamma_3 & 0 & 0 & 0 \\ 0 & 0 & \delta_4 & \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & \delta_5 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & \delta_6 & \gamma_6 \end{bmatrix} \vec{u_0} = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \\ b_{04} \\ b_{05} \\ b_{06} \end{bmatrix} . \tag{56}$$

Let the coefficient matrix of (56) be denoted M_0. Likewise we have a matrix equation for 6 equations

$$a_{1i} = (\alpha_i, \beta_i) = \begin{bmatrix} u_{1i} \\ u_{1,i-1} \end{bmatrix} ,$$

which is

$$\begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 & \beta_1 \\ \beta_2 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_3 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & \beta_4 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & \beta_5 & \alpha_5 & 0 \\ 0 & 0 & 0 & 0 & \beta_6 & \alpha_6 \end{bmatrix} \vec{u_1} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \end{bmatrix} . \tag{57}$$

Again let the coefficient matrix of (57) be denoted M_1. Then equation (54) becomes

$$M_0 u_0 = M_1 u_1 . \tag{58}$$

So the idea is to find two Weil divisors u_0 and u_1 satisfying (58). Using the computer algebra system Maple, we see that $\operatorname{rank} M_0 = \operatorname{rank} M_1 = 5$. Let z_j be a nonzero vector in $\ker M_j$. Then $\ker z_0 \cap \ker z_1 = \operatorname{colspace} M_0 \cap \operatorname{colspace} M_1$. Now $\ker \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ has rank 4 and $\ker M_0$ has rank 1 so the set of all possible u_0 satisfying (58) has rank 5. For each such u_0 there is a line of possible u_1 since $\ker M_1$ has rank 1. For each such pair (u_0, u_1) there is a potential function ϕ whose gradient satisfies

$$\vec{\nabla}\phi = h_0\vec{i} + h_1\vec{j} . \tag{59}$$

Of course ϕ is only unique up to additive constant.

The usual constraints on a portfolio are $c_0 + c_1 = 1$, $c_0 \geq 0$ and $c_1 \geq 0$. We also want to choose ϕ so that it is concave up, if not everywhere at least on the cones that overlap with these constraints. So we require that on σ_i the quadratic form for ϕ is positive definite for $i = 6, 1, 2$. In our

The support function h embeds into the group of Weil divisors by the rule

$$h \mapsto \begin{bmatrix} h(\eta_1) \\ \vdots \\ h(\eta_6) \end{bmatrix} \tag{49}$$

The Weil divisors of the $\vec{i}$ and $\vec{j}$ components of the scaled gradient of V are:

$$\vec{v_0} = (8, -6, -32, -8, 6, 32) \tag{50}$$

$$\text{and } \vec{v_1} = (18, 7, 10, -18, -7, -10)\ . \tag{51}$$

Check that these divisors are in the image of the function α, since they correspond to linear support functions. The Weil divisors of the $\vec{i}$ and $\vec{j}$ components of the scaled gradient of SV are:

$$s\vec{v}_0 = (80, -28, -112, 32, 8, 80) \tag{52}$$

$$\text{and } s\vec{v}_1 = (16, 19, 76, -92, -23, 16)\ . \tag{53}$$

Check that the images of these divisors are independent in $\mathrm{Cl}(X)$.

Because the 2 Weil divisors that arise from the variance function are linear, they correspond to free line bundles on X. Viewing the gradients as direct sums of support functions, we see that the variance defines a free vector bundle on X of rank 2. The 2 components of the gradient of the semivariance define 2 Weil divisors that are independent in $\mathrm{Cl}(X)$. Therefore we see that the semivariance defines a vector bundle of rank 2 on X that is a direct sum of 2 nontrivial vector bundles. The determinant of this direct sum is the element of $\mathrm{Pic}\, X$ that is the product of the 2 direct summands, hence is nontrivial.

We see that if ϕ is a risk function on (26), (27), that is locally quadratic on the fan in Figure 1, then the divergence of ϕ, $\vec{\nabla} \cdot \phi$ is a locally linear function on the fan of Figure 1 hence defines a line bundle in $\mathrm{Pic}\, X$. So up to determinants, we ask whether $\mathrm{Pic}\, X \otimes \mathbb{Q}$ parametrizes a large class of possible risk functions ϕ on the time series (26), (27). That is, how large a subset of $\mathrm{Pic}\, X \otimes \mathbb{Q}$ consists of vector bundles $\mathfrak{L}$ that are the determinant of a locally quadratic risk function? For the present example, we can show that there is a convex subset of $\mathrm{Pic}\, X \otimes \mathbb{Q}$ of dimension 3 with this property.

The idea is to find 2 Weil divisors

$$\vec{u_0} = (u_{01}, \ldots, u_{06}) \text{ and } \vec{u_1} = (u_{11}, \ldots, u_{16})$$

that define 2 Δ-linear support functions h_0 and h_1 which satisfy

$$\frac{\partial h_0}{\partial c_1} = \frac{\partial h_1}{\partial c_0} \tag{54}$$

The support function h_j is defined on the cone σ_i by the equations $h_0(\eta_i) = u_{0i}$. So if $h_j|_{\sigma_i} = a_{ji}c_0 + b_{ji}c_1$, then h_j is defined by the matrix equations

$$\begin{bmatrix} \eta_i \\ \eta_{i-1} \end{bmatrix} \begin{bmatrix} a_{ji} \\ b_{ji} \end{bmatrix} = \begin{bmatrix} u_{ji} \\ u_{j,i-1} \end{bmatrix}\ . \tag{55}$$

problems can be solved if one knows the gradient functions $\vec{\nabla}\,\mathrm{V}$, $\vec{\nabla}\,\mathrm{SV}$ and any potential functions ϕ, ψ such that $\vec{\nabla}\,\mathrm{V} = \vec{\nabla}\phi$ and $\vec{\nabla}\,\mathrm{SV} = \vec{\nabla}\psi$. So in this sense, the important functions are the gradients, not the the risk functions themselves. We emphasize this because the gradients are piecewise linear forms on the (c_0, c_1)-plane. As we are about to see, such functions define vector bundles on the toric variety associated to the fan shown in Figure 1.

Because the sequences (24) (25) are integral, the equations (37), (38) and (39) are defined over the rational numbers $\mathbb{Q}$. Following [7], [3] we call the regions σ_i strongly convex rational polyhedral cones. The set Δ of all intersections $\sigma_i \cap \sigma_j$ is called a fan on $\mathbb{R}^2$. Associated to the fan Δ is a variety $X = T_N\,\mathrm{emb}(\Delta)$ defined over the complex numbers $\mathbb{C}$. In general X will be complete, rational, integral, normal but singular. The singularities are always rational. In our example, X will be a surface with 6 singular points. Built into the toric variety are many of the arithmetical properties of the 3 lines L_1, L_2, L_3. For example, the 3 lines L_1, L_2, L_3 define 6 rays emanating from the origin. On each of these 6 rays is a unique lattice point (n_0, n_1) such that $\gcd(n_0, n_1) = 1$. In our example, these 6 lattice points are $\eta_1 = (19, 28)$, $\eta_2 = (-1, 5)$, $\eta_3 = (-23, -8)$, $\eta_4 = (-19, -28)$, $\eta_5 = (1, -5)$, $\eta_6 = (23, 8)$. Of course these 6 points completely determine the fan Δ. Many of the important invariants of $X = T_N\,\mathrm{emb}(\Delta)$ can be computed in terms of these 6 points (see [1], [2]). For example, the Picard group of X is $\mathrm{Pic}\,X \cong \mathbb{Z}^4$, the class group of Weil divisors is $\mathrm{Cl}\,X \cong \mathbb{Z}^4 \oplus \mathbb{Z}/123$. The Brauer group of X is $\mathrm{B}(X) \cong \mathbb{Z}/123$.

A Δ-linear support function (or support function, for short) is a real-valued function defined on $\mathbb{R}^2$ which is a linear functional upon restriction to any of the cones σ_i and which is integer valued on the points $\eta_1, \ldots, \eta_6$. The gradients $\vec{\nabla}\,\mathrm{V}$ and $\vec{\nabla}\,\mathrm{SV}$ are functions which are linear on each σ_i, but they are vector-valued, not real-valued and they are rational valued, not integral, on the η_i. Therefore, they can be scaled to integral vector-values by the appropriate rational numbers. Check that $162/41$ works for $\vec{\nabla}\mathrm{SV}$ and $27/41$ works for $\vec{\nabla}\mathrm{V}$. Scaling the gradients is within the allowable framework of minimizing the risk functions. That is, it does not affect where the risk functions are minimized. The $\vec{i}$ and $\vec{j}$ components of these scaled gradients are each support functions.

The set of all support functions on Δ is a free abelian group of rank 6. A support function is completely determined by its values on the 6 points $\eta_1, \ldots, \eta_6$. There is an exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^6 \longrightarrow \mathrm{Cl}(X) \longrightarrow 0 \ . \tag{47}$$

The center group in (47) is called the group of T_N-invariant Weil divisors. The function α has matrix

$$\begin{bmatrix} \eta_1 \\ \vdots \\ \eta_6 \end{bmatrix} . \tag{48}$$

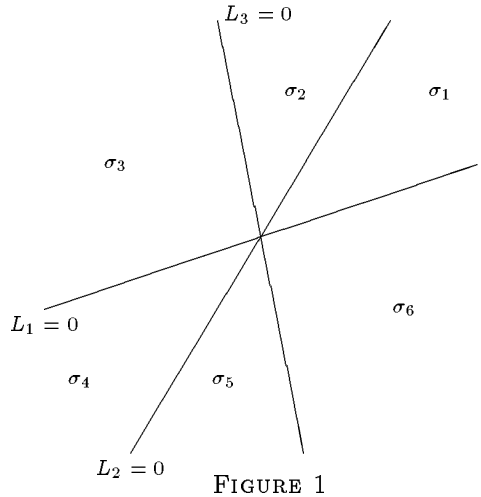

FIGURE 1

The 3 equations

$$L_1 = -4/9c_0 + 23/18c_1 = 0 \tag{42}$$

$$L_2 = 14/9c_0 - 19/18c_1 = 0 \tag{43}$$

$$L_3 = -10/9c_0 - 2/9c_1 = 0 \tag{44}$$

partition the (c_0, c_1)-plane into 6 regions. Label these regions $\sigma_1, \ldots, \sigma_6$ as shown in Figure 1. The function SV is a piecewise quadratic on the (c_0, c_1)-plane which is a quadratic form on each σ_i.

$$\mathrm{SV}(c_0, c_1) = \begin{cases} L_3^2/3 & \text{on} \quad \sigma_1 \\ (L_2^2 + L_3^2)/3 & \text{on} \quad \sigma_2 \\ L_2^2/3 & \text{on} \quad \sigma_3 \\ (L_1^2 + L_2^2)/3 & \text{on} \quad \sigma_4 \\ L_1^2/3 & \text{on} \quad \sigma_5 \\ (L_1^2 + L_3^2)/3 & \text{on} \quad \sigma_6 \end{cases} \tag{45}$$

The gradient of the semivariance function is a piecewise linear transformation which is linear on each σ_i.

(46) $\vec{\nabla}\,\mathrm{SV}(c_0, c_1) =$

$$\begin{cases} (200/243c_0 + 40/243c_1)\vec{i} + (40/243c_0 + 8/243c_1)\vec{j} & \text{on } \sigma_1 \\ (592/243c_0 - 226/243c_1)\vec{i} + (-226/243c_0 + 377/486c_1)\vec{j} & \text{on } \sigma_2 \\ (392/243c_0 - 226/243c_1)\vec{i} + (-226/243c_0 + 361/486c_1)\vec{j} & \text{on } \sigma_3 \\ (424/243c_0 - 358/243c_1)\vec{i} + (-358/243c_0 + 445/243c_1)\vec{j} & \text{on } \sigma_4 \\ (32/243c_0 - 92/243c_1)\vec{i} + (-92/243c_0 + 529/486c_1)\vec{j} & \text{on } \sigma_5 \\ (232/243c_0 - 52/243c_1)\vec{i} + (-52/243c_0 + 545/486c_1)\vec{j} & \text{on } \sigma_6 \end{cases}$$

In applications (see [5], [6]) one seeks to minimize one of the risk functions V or SV subject to some constraints such as $c_0 + c_1 = 1$, and $0 \leq c_0, c_1 \leq 1$. By the method of Lagrange multipliers, these optimization expectation of the portfolio (c_0, c_1) is $\mathrm{E}(c_0, c_1) = c_0\mu^{(0)} + c_1\mu^{(1)} = 4/9c_0 - 13/18c_1$. The variance of the portfolio (c_0, c_1) is defined to be

$$\mathrm{V}(c_0, c_1) = \frac{1}{3}\sum_{k=1}^{3}\left[c_0 r_k^{(0)} + c_1 r_k^{(1)} - \mathrm{E}(c_0, c_1)\right]^2 \tag{28}$$

$$= \frac{1}{3}\sum_{k=1}^{3}\left[c_0(r_k^{(0)} - \mu^{(0)}) + c_1(r_k^{(1)} - \mu^{(1)})\right]^2 \tag{29}$$

Set

$$A = \frac{1}{3}\sum_{k=1}^{3}\left(r_k^{(0)} - \mu^{(0)}\right)^2 = 104/81 \tag{30}$$

$$B = \frac{1}{3}\sum_{k=1}^{3}\left(r_k^{(0)} - \mu^{(0)}\right)\left(r_k^{(1)} - \mu^{(1)}\right) = -53/81 \tag{31}$$

$$C = \frac{1}{3}\sum_{k=1}^{3}\left(r_k^{(1)} - \mu^{(1)}\right)^2 = 151/162 \tag{32}$$

Now (28) becomes

$$\mathrm{V}(c_0, c_1) = Ac_0^2 + 2Bc_0c_1 + Cc_1^2 = 104/81c_0^2 - 106/81c_0c_1 + 151/162c_1^2 \tag{33}$$

That is, V is a quadratic form. The gradient of V is a linear transformation

$$\vec{\nabla}\,\mathrm{V}(c_0, c_1) = (2Ac_0 + 2Bc_1)\vec{i} + (2Bc_0 + 2Cc_1)\vec{j} \tag{34}$$

$$= (208/81c_0 - 106/81c_1)\vec{i} + (-106/81c_0 + 151/81c_1)\vec{j} \tag{35}$$

For $k = 1, 2, 3$ define

$$L_k(c_0, c_1) = c_0(r_k^{(0)} - \mu^{(0)}) + c_1(r_k^{(1)} - \mu^{(1)}) \tag{36}$$

Then

$$L_1 = -4/9c_0 + 23/18c_1 \tag{37}$$

$$L_2 = 14/9c_0 - 19/18c_1 \tag{38}$$

$$L_3 = -10/9c_0 - 2/9c_1 \tag{39}$$

The semivariance function of the portfolio (c_0, c_1) is

$$\mathrm{SV}(c_0, c_1) = \frac{1}{3}\sum_{k=1}^{3}\left(L_k(c_0, c_1)\right)^2 \epsilon_k \tag{40}$$

where

$$\epsilon_k = \begin{cases} 0 & \text{if } L_k(c_0, c_1) \geq 0 \\ 1 & \text{otherwise} \end{cases} . \tag{41}$$

$$\vec{\nabla}\,\mathrm{SV}(\vec{c}) = \left(\frac{\partial}{\partial c_0}\,\mathrm{SV}(\vec{c}),\ldots,\frac{\partial}{\partial c_{N-1}}\,\mathrm{SV}(\vec{c})\right) \text{ , where} \tag{19}$$

$$\frac{\partial}{\partial c_\nu}\,\mathrm{SV}(\vec{c}) = \frac{1}{l-1}\sum_{k=1}^{l} 2L_k(\vec{c})\left(r_k^{(\nu)} - \mu^{(\nu)}\right)\epsilon_k \text{ , and where} \tag{20}$$

$$\epsilon_k = \begin{cases} 0 & \text{if } L_k(\vec{c}) \geq 0 \\ 1 & \text{otherwise} \end{cases} . \tag{21}$$

Each partial derivative (18) defines a Δ-linear support function on $\mathbb{R}^N$. Unfortunately, $\frac{\partial}{\partial c_\nu}\,\mathrm{V}(\vec{c})$ is not integral valued on $\mathbb{Z}^N$. However it is rational valued, so for some scaling factor s which can be chosen to work for all ν, we can view $s\frac{\partial}{\partial c_\nu}\,\mathrm{V}(\vec{c})$ as an element of M. Because $\vec{\nabla}\,\mathrm{V}$ is the direct sum of N linear functionals, under the map (16) we associate to the variance function a direct sum of N copies of the trivial line bundle. Similarly, for some scaling factor t, $t\frac{\partial}{\partial c_\nu}\,\mathrm{SV}(\vec{c})$ is integral valued on $\mathbb{Z}^N$. Again under (16) there is associated to $t\frac{\partial}{\partial c_\nu}\,\mathrm{SV}(\vec{c})$ a line bundle $\mathcal{L}(\mathrm{SV}_\nu)$. Viewing the gradient $\vec{\nabla}\,\mathrm{SV}$ as a direct sum of its N partial derivatives, we associate to the semivariance function a direct sum of N line bundles

$$\mathcal{L}(\mathrm{SV}_0) \oplus \cdots \oplus \mathcal{L}(\mathrm{SV}_{N-1}) . \tag{22}$$

As we see in the next section, the line bundles in this direct sum can be non-trivial and independent in $\operatorname{Pic} X$.

We could also consider the divergence $\vec{\nabla}\cdot\mathrm{SV}$ function, which is a function which is locally linear on Δ. As above, after scaling by t, $\vec{\nabla}\cdot t\,\mathrm{SV} = \sum_{\nu=0}^{N-1} t\frac{\partial}{\partial c_\nu}\,\mathrm{SV}(\vec{c})$ is a support function in $\mathcal{SF}(\Delta)$. Because the map to $\operatorname{Pic} X$ in (16) is a group homomorphism, the divergence of SV maps to the product of the line bundles defined above

$$\mathcal{L}(\mathrm{SV}_0) \otimes \cdots \otimes \mathcal{L}(\mathrm{SV}_{N-1}) . \tag{23}$$

This product is again a line bundle, in fact it is the determinant, or highest exterior power of (22).

4. An Example

Let us consider a simple example. We take 2 time series of 4 integers each.

$$\vec{P}_0 = (p_i^{(0)}) = (1,1,3,1) \tag{24}$$

$$\vec{P}_1 = (p_i^{(1)}) = (1,3,2,3) \tag{25}$$

Suppose these represent prices of securities. Construct the corresponding sequences of daily returns.

$$\vec{R}_0 = (r_i^{(0)}) = (0,2,-2/3) \tag{26}$$

$$\vec{R}_1 = (r_i^{(1)}) = (2,-1/3,1/2) \tag{27}$$

For (26) and (27) compute the means: $\mu^{(0)} = 4/9$, $\mu^{(1)} = -13/18$. A portfolio associated to (24) and (25) is a pair of numbers (c_0, c_1). The singularities of X are always rational. The N-dimensional algebraic torus $T_N = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ acts on $T_N \operatorname{emb}(\Delta)$ and the orbits under the action correspond to the cones σ in the fan Δ.

We deviate slightly from the notation of [7] and denote by $\mathbb{Z}^N$ the set of points in $\mathbb{R}^N$ with integral coordinates. A Δ-linear support function (or support function, for short) is a real-valued function defined on $\mathbb{R}^N$ which is a linear functional upon restriction to any of the cones $\sigma \in \Delta$ and which is integer valued on the points in $\mathbb{Z}^N$. The set of all support functions on Δ make up a free abelian group of finite rank which we denote by $\mathcal{SF}(\Delta)$. The subgroup consisting of all linear support functions is usually denoted M and is isomorphic to $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^N, \mathbb{Z})$, hence is free of rank N. We view elements y in M as N-tuples $y = (m_0, \ldots, m_{N-1})$ that act on $\mathbb{R}^N$ by inner product. The linear functionals $y \in M$ are identified with monomials $e(y) \in \mathbb{C}[x_0, x_0^{-1}, \ldots, x_{N-1}, x_{N-1}^{-1}]$ by the map which takes $(m_0, \ldots, m_{N-1})$ to $\prod_{\nu=0}^{N-1} x_\nu^{m_\nu}$.

The toric variety X has a canonical covering by open affine sets $\{U_\sigma | \sigma \in \Delta\}$. The sheaf of regular functions $\mathcal{O}_X$ is defined as follows. On U_σ we have $\mathcal{O}_X(U_\sigma)$ is the subring of $\mathbb{C}[x_0, x_0^{-1}, \ldots, x_{N-1}, x_{N-1}^{-1}]$ generated by $\{e(y) | y \cdot x \geq 0 \text{ for all } x \in \sigma\}$.

As shown in [7] there is an exact (split-exact if Δ is complete, or nearly so) sequence of abelian groups

$$0 \longrightarrow M \longrightarrow \mathcal{SF}(\Delta) \longrightarrow \operatorname{Pic} X \longrightarrow 0 \ . \tag{16}$$

Which is defined on elements by taking a support function h to its associated line bundle $\mathcal{L}_h$. So if Δ is complete, $\operatorname{Pic} X$ is a free abelian group. The rank of $\operatorname{Pic} X$ can be as low as 0 or as large as $\dim_{\mathbb{Q}} \operatorname{Cl}(X) \otimes \mathbb{Q} = n - N$, where n is the cardinality of the set $\{\sigma \in \Delta | \dim \sigma = 1\}$.

Let us briefly describe the line bundle $\mathcal{L}_h$ as a locally free sheaf on X of rank 1. The sheaf $\mathcal{L}_h$ is free on each U_σ and is viewed as a subsheaf of the constant sheaf $V \mapsto K$, K being the quotient field of X. The support function h is linear on each cone $\sigma \in \Delta$. Therefore, we can think of h as a set of elements of M: $\{y_\sigma \in M | \text{ for each } \sigma, \tau \in \Delta, y_\sigma|_\tau = y_\tau|_\sigma\}$. The generator for $\mathcal{L}_h$ on U_σ is $e(-y_\sigma)$.

As examples of Δ-linear support functions, consider the gradients of V and SV .

$$\vec{\nabla}\,\mathrm{V}(\vec{c}) = \left(\frac{\partial}{\partial c_0}\,\mathrm{V}(\vec{c}), \ldots, \frac{\partial}{\partial c_{N-1}}\,\mathrm{V}(\vec{c})\right) \ , \text{ where} \tag{17}$$

$$\frac{\partial}{\partial c_\nu}\,\mathrm{V}(\vec{c}) = \frac{1}{l-1} \sum_{k=1}^{l} 2L_k(\vec{c}) \left(r_k^{(\nu)} - \mu^{(\nu)}\right) \ . \tag{18}$$

subject to the constraints

$$\mathrm{E}(\vec{c}) = \sum_{\nu=0}^{N-1} c_\nu \mu^{(\nu)} \geq E_0,\ 0 \leq c_\nu \leq 1,\ \sum_{\nu=0}^{N-1} c_\nu = 1, \tag{14}$$

where E_0 is the investor's expectation. That is, E_0 is a fixed constant, determined by the investor's risk tolerance.

One can show that $\mathrm{SV}(\vec{c})$ is a continuously differentiable function of the variables $c_0,\ c_1, \ldots,\ c_{N-1}$. By [8, Theorem 9.16] this implies that $\mathrm{SV}(\vec{c})$ is continuously differentiable as a function of $\vec{c}$. It is even locally quadratic and convex in the control space of the c_k's. Examples show that the function $\mathrm{SV}(\vec{c})$ is C^1, but not C^2.

3. The Toric Variety of a Portfolio

To the sequences of integers (1) we will associate an algebraic variety and to the functions (10) and (12) we associate vector bundles on this variety. The variety associated to (1) is a toric variety. As a general reference for the theory of toric varieties, we refer to [7] or [3] and for Algebraic Geometry to [4]. A toric variety is completely determined by an arrangement of convex sets in $\mathbb{R}^r$ called a fan. In our case, the fan will be determined by the linear equations

$$L_k(\vec{c}) = 0\ ,\ (1 \leq k \leq l)\ , \tag{15}$$

where L_k is as in (11). For each k, $L_k(\vec{c}) = 0$ defines a hyperplane through the origin in $\mathbb{R}^N$.

The equations (15) partition $\mathbb{R}^N$ into convex polyhedral sets. These convex sets are cones with vertices at the origin, meaning that if they contain a vector $\vec{x}$, they contain the ray emanating from the origin containing $\vec{x}$. First notice that if l is sufficiently large and the sequences (1) are sufficiently general, the linear functions (11) will be non-zero and the set of hyperplanes defined by (15) will be in "general position". This means that each cone defined by the hyperplanes will be strongly convex. Roughly this means that the angle of the vertex of each cone is less than 180 degrees, or that the cone contains no line through the origin. Because the prices (1) are assumed to be integers, the equations (15) are defined over the field of rational numbers $\mathbb{Q}$. So the cones cut out by the equations are, in the common terminology, strongly convex rational polyhedral cones.

The set of all N-dimensional cones σ cut out by (15) together with all lower dimensional faces of these cones make up what is called a fan. Correctly speaking, a fan Δ consists of a nonempty collection of strongly convex rational polyhedral cones satisfying: (a) if $\sigma \in \Delta$, then every face of σ is in Δ and (b) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a common face of σ and τ. The fan defined by (15) is moreover complete since every point of $\mathbb{R}^N$ is in some cone of the fan.

Associated to a fan Δ is a toric variety $X = T_N \operatorname{emb}(\Delta)$. The variety X is defined over the complex number field $\mathbb{C}$. In general $T_N \operatorname{emb}(\Delta)$ is rational, integral, normal but singular and complete if Δ is complete. The and the *semivariance* of $\vec{R_\nu}$ as

$$\text{SV}^{(\nu)} = \frac{1}{l-1}\sum_{k=1}^{l}(r_k^{(\nu)} - \mu^{(\nu)})^2\epsilon_k, \tag{6}$$

where

$$\epsilon_k = \begin{cases} 0 & \text{if} \quad r_k^{(\nu)} - \mu^{(\nu)} \geq 0 \\ 1 & \text{if} \quad r_k^{(\nu)} - \mu^{(\nu)} < 0 \end{cases}.$$

Let $\vec{R_0}, \vec{R_1}, \ldots, \vec{R_{N-1}}$ be sequences of daily returns as in (2). Let $\vec{c} = (c_0, c_1, \ldots, c_{N-1})$ be a vector. In applications, we usually assume there are some constraints on $\vec{c}$ such as, for each ν, c_ν satisfies $0 \leq c_\nu \leq 1$ and $\sum_{\nu=0}^{N-1} c_\nu = 1$. For now, we do consider such constraints. We define a *portfolio* to be a formal expression

$$\text{PF}(\vec{c}) = \langle c_0, c_1, \ldots, c_{N-1}\rangle. \tag{7}$$

For a portfolio $\text{PF}(\vec{c})$ there is the *associated time series of daily returns*

$$\text{TS}(\text{PF}(\vec{c})) = \text{TS}(\vec{c}) = (\sum_{\nu=0}^{N-1} c_\nu r_1^{(\nu)}, \ldots, \sum_{\nu=0}^{N-1} c_\nu r_n^{(\nu)}, \ldots, \sum_{\nu=0}^{N-1} c_\nu r_l^{(\nu)}). \tag{8}$$

We define the *expectation* of the portfolio $\text{PF}(\vec{c})$ to be

$$\text{E}(\vec{c}) = \sum_{\nu=0}^{N-1} c_\nu(\text{expectation of } \vec{R_\nu}) = \sum_{\nu=0}^{N-1} c_\nu \mu^{(\nu)}. \tag{9}$$

The variance of $\text{PF}(\vec{c})$ is defined to be

$$\text{V}(\vec{c}) = \frac{1}{l-1}\sum_{k=1}^{l}\left[\sum_{\nu=0}^{N-1} c_\nu r_k^{(\nu)} - \text{E}(\vec{c})\right]^2. \tag{10}$$

For each k from 1 to l define the linear function

$$L_k(\vec{c}) = \sum_{\nu=0}^{N-1} c_\nu r_k^{(\nu)} - \text{E}(\vec{c}) = \sum_{\nu=0}^{N-1} c_\nu \left(r_k^{(\nu)} - \mu^{(\nu)}\right). \tag{11}$$

Define the *semivariance* as in (6)

$$\text{SV}(\vec{c}) = \frac{1}{l-1}\sum_{k=1}^{l}\left[L_k(\vec{c})\right]^2 \epsilon_k, \tag{12}$$

where

$$\epsilon_k = \begin{cases} 0 & \text{if } L_k(\vec{c}) \geq 0 \\ 1 & \text{otherwise} \end{cases}.$$

The basic philosophy that was inspired by Markowitz in [5] is to minimize $\text{V}(\vec{c})$ or $\text{SV}(\vec{c})$ with respect to the variable $\vec{c}$ and a given range of expected returns. Therefore, the problem is

$$\text{to minimize } f(\vec{c}) = \text{V}(\vec{c}) \text{ (or } \text{SV}(\vec{c})) \tag{13}$$

The purpose of introducing the algebraic variety is to try to apply the machinery of Algebraic Geometry to problems arising in the Theory of Mathematics of Finance. This note answers few if any such questions and raises more than it answers. The rule which is introduced later that assigns to a set of securities a toric variety raises many questions. For example, is there a functor behind it all? If so, on what category is it defined? One reason for looking at the vector bundles associated to the risk functions is to see if there is a systematic way of classifying all risk functions on the initial set of securities. So one could ask whether there are other vector bundles of rank N on the variety that come from risk functions. If so, do they provide better measures, or do they provide the investor with efficient portfolios with increased yields?

Why map the risk functions to direct sums of line bundles on a toric variety? One sensible reason is that the set of line bundles on a toric variety is a free abelian group of finite rank. One could try to consider the set of all risk functions. This would be an overwhelmingly large set of functions. By mapping the risk functions into a smaller group, there is hope that new information can be obtained due to the sparsity of trees in the forest.

2. The Risk Functions and Portfolio Selection Methods

Given are sequences of daily *prices* on N securities

$$\vec{P}_\nu = (p_0^{(\nu)}, p_1^{(\nu)}, \ldots, p_n^{(\nu)}, p_{n+1}^{(\nu)}, \ldots, p_l^{(\nu)}) \tag{1}$$

where $0 \leq \nu \leq N-1$. For our purposes it will be useful to assume the prices are positive integers. From (1) we construct the corresponding sequences of daily *returns*

$$\vec{R}_\nu = (r_1^{(\nu)}, r_2^{(\nu)}, \ldots, r_n^{(\nu)}, r_{n+1}^{(\nu)}, \ldots, r_l^{(\nu)}) \tag{2}$$

where $0 \leq \nu \leq N-1$ and $r_{n+1}^{(\nu)}$ is defined by

$$r_{n+1}^{(\nu)} = \frac{p_{n+1}^{(\nu)} - p_n^{(\nu)}}{p_n^{(\nu)}}. \tag{3}$$

For each ν $(0 \leq \nu \leq N-1)$ we define the *sample mean* $\bar{r^{(\nu)}}$ of $\vec{R}_\nu$ as usual

$$\bar{r^{(\nu)}} = \mu^{(\nu)} = \frac{1}{l}\sum_{k=1}^{l} r_k^{(\nu)}. \tag{4}$$

(Note $\mu^{(\nu)}$ is also called the *expected value* of $\vec{R}_\nu$.) We define the *sample variance* of $\vec{R}_\nu$ as

$$(\sigma^{(\nu)})^2 = \mathrm{V}^{(\nu)} = \frac{1}{l-1}\sum_{k=1}^{l}(r_k^{(\nu)} - \mu^{(\nu)})^2, \tag{5}$$

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A Variety Associated to a Portfolio

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Abstract

In the portfolio selection theory of H. Markowitz one is led to consider the variance function defined on a set of time series. Also one may consider the semivariance function. We define for each such set of time series a toric variety and show that the risk functions mentioned by Markowitz correspond to vector bundles on the variety. The variance corresponds to a free bundle and semivariance to a direct sum of non-free line bundles.

1. Introduction

If one has a certain amount of assets and wants to allocate them among N securities, the philosophy of Markowitz is to diversify efficiently. The idea is to pick the portfolio which minimizes the investor's risk. The measure of risk is an important factor to consider. In this note we consider 2 of Markowitz's risk functions: the variance and semivariance functions. The philosophy behind using the variance function as a risk function is that one should minimize the deviation from the mean. The semivariance method says that positive deviation from the mean is okay, but negative deviation should be minimized.

We associate to a set of prices of N securities a toric variety and to the risk functions vector bundles of rank N (i.e. locally free sheaves of rank N) on this variety. The variance function corresponds to the free vector bundle of rank N, and the semivariance function to a direct sum of N line bundles (i.e. invertible modules, or rank-1 vector bundles).

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