4. M. Raynaud, *Anneaux Locaux Henséliens*, vol. 169 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1970.

z into $\hat{A} = k[x][[y]]$, the completion of $k[x, y]$ with respect to the (y)-adic topology. $\square$

Corollary 5. *Let $\phi(z) \in k[x, y][z]$ be an irreducible polynomial with leading coefficient $p = p(x, y)$. Assume $p(x, 0) \in k^*$ (that is, $p(x, y)$ is of the form $yp_1(x, y) + \lambda$ for some $\lambda \in k^*$. Let C be the zero set of the discriminant of $\phi(z)$ and $\phi'(z)$. Suppose C is disjoint from the line $y = 0$. Then the residue of z in the ring $k[x, y][z]/(\phi(z))$ satisfies an equation of the form*

$$z = \sum_{i=0}^{\infty} \alpha_i(x) y^i,$$

where $\alpha_i(x) \in k[x]$.

Proof. Let $A_1 = k[x, y][1/p(x, y)]$, $\phi_1(z) = \phi(z)/p \in A_1[z]$ and $B_1 = A_1[z]/(\phi_1(z))$. Then $X_1 = \operatorname{Spec} A_1$ is an open affine neighborhood of the line $y = 0$ in $X = \operatorname{Spec} A$. The map $f\colon Y_1 \to X_1$ is finite and faithfully flat since $\phi_1(z)$ is monic. Proceed as in the proof of Proposition 4, with X_1 and Y_1 replacing X and Y. The conclusion is that z maps into $\hat{A}_1$, the completion of A_1 with respect to the (y)-adic topology. But $p(x, 0) = \lambda \in k^*$ so $\hat{A}_1 = \hat{A}$ (see [2, section 24]). $\square$

Lastly we consider the special case where ω satisfies an equation of the form $z^n = P(x, y)$ where $P(x, y) \in k[x, y]$ is a square-free polynomial.

Proposition 6. *Let $P(x, y) \in k[x, y]$ be a square-free polynomial and assume $P(0, 0) \neq 0$. Let $\omega \in k[[x, y]]$ satisfy $\omega^n = P(x, y)$. Then $\omega \in k[x][[y]]$ if and only if $P(x, 0) \in k^*$ (that is, $P(x, 0)$ is of the form $yP_1(x, y) + \lambda$, for some $\lambda \in k^*$).*

Proof. Set $A = k[x, y]$, $B = A[z]/(z^n - P(x, y))$, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. If the branch locus of $f\colon Y \to X$ is R, then $f(R) = C$ is the zero set of the polynomial $P(x, y)$. If $P(x, y) = yP_1(x, y) + \lambda$, for some $\lambda \in k^*$, then the branch locus C is disjoint from the line $y = 0$, so by Proposition 4 we can map z to $\omega \in k[x][[y]]$. Conversely, suppose the curves C and $y = 0$ meet at some point q. Then for some $\kappa \in k$, the maximal ideal of A associated to q is $M = (x - \kappa, y) = I(q)$ and $P = P(x, y) \in M$. Let $\mathcal{O}_q$ be the local ring of X at q and $\hat{\mathcal{O}}_q$ the completion. Then $P \in \hat{M}$. Since the extension $\hat{\mathcal{O}}_q/\mathcal{O}_q$ is unramified, P remains square-free in $\hat{\mathcal{O}}_q$. Therefore $z^n - P(x, y)$ is irreducible in $\hat{\mathcal{O}}_q[z]$. But we can view $k[x][[y]] = k[x - \kappa][[y]]$ as a subring of $\hat{\mathcal{O}}_q = k[[x - \kappa, y]]$. So if $\omega \in k[x][[y]]$, then $z^n - P(x, y)$ is not irreducible in $\hat{\mathcal{O}}_q[z]$. $\square$

References

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for ω to be in $\hat{A} = k[x][[y]]$ when ω satisfies an irreducible polynomial $\phi(z) \in A[z] = k[x,y][z]$. It turns out that it is sufficient for $A \to B$ to be unramified along $y = 0$. The reason for this is the line $y = 0$ is simply connected (k is algebraically closed).

Proposition 4. *Let $\phi(z) \in k[x,y][z]$ be a monic irreducible polynomial. Let C be the zero set of the discriminant of $\phi(z)$ and $\phi'(z)$. Suppose C is disjoint from the line $y = 0$. Then the residue of z in the ring $k[x,y][z]/(\phi(z))$ satisfies an equation of the form*

$$z = \sum_{i=0}^{\infty} \alpha_i(x) y^i,$$

where $\alpha_i(x) \in k[x]$.

Proof. Let $A = k[x,y]$ and $B = A[z]/(\phi(z))$. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $f: Y \to X$ the structure map. The *different* $\delta_{Y/X}$ of Y over X is the annihilator of $\Omega^1_{Y/X}$ and is an ideal in $\mathcal{O}_Y$. The closed subscheme R of Y defined by $\delta_{Y/X}$ is called the *branch locus* of Y over X. The branch locus is the set of zeros of $\phi'(z)$ [3, p. 22]. Since B is a free A-module of rank $\deg \phi$, f is finite and faithfully flat. By the Theorem of the Purity of the Branch Locus [3, p. 24], if the branch locus is non-empty, it is of pure codimension 1 in Y. Let $R \subseteq Y$ denote the branch locus and $f(R) = C$ the image of R in X. Then by elimination theory, C is the curve on X defined by the discriminant of $\phi(z)$ and $\phi'(z)$. Let $D \subseteq X$ be a connected curve and suppose $C \cap D = \emptyset$. Therefore upon restriction to $f^{-1}(D)$, f is an étale morphism. Assume that the étale cover $f^{-1}(D) \to D$ has a section $\sigma: D \to f^{-1}(D)$. That is, that $f^{-1}(D) = D_1 + D_2$ where D_1 is a component isomorphic to D. Set $U = X - C$ and $V = Y - R - D_2$. Then U and V are Zariski neighborhoods of D and D_1 respectively. Thus $f: V \to U$ is an étale neighborhood of D. If we let $(\tilde{U}, \tilde{D})$ denote the henselization of the couple (U, D), then we have the following commutative diagram of couples [4].

$$\begin{array}{ccc} (V, D_1) & \stackrel{\eta}{\leftarrow} & (\tilde{U}, \tilde{D}) \\ \downarrow & \swarrow & \\ (U, D) & & \end{array}$$

We can view $z \in B$ as a rational function on Y and hence on V. By the morphism η, z restricts to a rational function on $\tilde{U}$, hence is in the function field $K(\tilde{U})$. Of course this all depends on the choice of the section σ. But $\tilde{A} = \Gamma(\tilde{U}, \mathcal{O})$ is contained in the completion $\hat{A}$ of A with respect to the $I(D)$-adic topology. Since A is a unique factorization domain, $I(D)$ is a principal ideal. Say $I(D)$ is generated by Q. So there is a surjection $A[[w]] \to \hat{A}$ defined by sending w to Q [2]. Thus we can write z as a power series $z = \sum \alpha_i Q^i$ where $\alpha_i \in A$. To finish the proposition, take D to be the line $y = 0$. Then D is simply connected (k is algebraically closed) and $f^{-1}(D) = D_1 + \cdots + D_n$ where each D_i is isomorphic to D. So there are n choices for the section σ. For each choice of σ, we can map

Proof. The idea is to show that $A \longrightarrow B$ localizes to an étale neighborhood of p. The reason we can do this is $A \longrightarrow B$ is unramified over p and p is simply connected (k is algebraically closed).

Since f is unramified over p, there is an affine open neighborhood U of p such that if $V = f^{-1}(U)$, then $f: V \longrightarrow U$ is étale and quasi-finite. So $f^{-1}(p) = \{p_1, p_2, \ldots, p_d\}$. Let $V_1 = V - p_2 - \cdots - p_d$. Then $f: V_1 \longrightarrow U$ is an étale neighborhood of $p = (0,0)$. Therefore V_1 maps to the henselization of the local ring at p, making a commutative diagram.

$$(1) \qquad \begin{array}{ccc} \operatorname{Spec} \mathcal{O}_p^h & \longleftarrow & V_1 \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathcal{O}_p & \longleftarrow & U \end{array}$$

So $z \in \Gamma(V_1, \mathcal{O}_{V_1})$ is mapped to some $\omega \in \mathcal{O}_p^h \subseteq \hat{\mathcal{O}}_p = k[[x,y]]$. The commutative diagram

$$(2) \qquad \begin{array}{ccccc} B \subseteq & \operatorname{Spec} \mathcal{O}_{V,p_1} & \subseteq & \operatorname{Spec} \mathcal{O}_{V,p_1}^h \\ \uparrow & \uparrow & & \uparrow = \\ A \subseteq & \operatorname{Spec} \mathcal{O}_{U,p} & \subseteq & \operatorname{Spec} \mathcal{O}_{U,p}^h \end{array}$$

shows that the mapping $B \longrightarrow k[[x,y]]$ is an embedding. Notice that there are d choices for the embedding, one for each of the points p_i lying over p. □

The henselization of $X = \operatorname{Spec} k[x,y]$ along the subvariety Z corresponding to the ideal $I = (y)$ is the direct limit $\tilde{X} = \operatorname{Spec} \tilde{A}$ of all étale neighborhoods of Z. Since $\hat{A} = k[x][[y]]$ is the completion of $A = k[x,y]$ with respect to the I-adic topology, the pair $(\hat{A}, \hat{I})$ is a Hensel couple. Given any intermediate ring R such that $k[x,y] \subseteq R \subseteq k[[x,y]]$ and such that $\operatorname{Spec} R \longrightarrow X$ is an étale neighborhood of Z, there is a $k[x,y]$-homomorphism $R \longrightarrow k[x][[y]]$ which fixes x and y. So $R \subseteq k[x][[y]]$.

Proposition 3. *Let $\omega \in k[[x,y]]$ be algebraic over $A = k[x,y]$. Then ω is in $k[x][[y]]$ if and only if there is an intermediate ring R such that $k[x,y] \subseteq R \subseteq k[[x,y]]$, $\omega \in R$ and $k[x,y] \longrightarrow R$ is an étale neighborhood of $Z = \operatorname{Spec} k[x,y]/(y)$ in $\operatorname{Spec} k[x,y]$.*

Proof. If $A \longrightarrow R$ is an étale neighborhood of $Z = \operatorname{Spec} A/(y)$, then R maps onto a subring of $\hat{A} = k[x][[y]]$ by a $k[x,y]$-homomorphism. Since $k[x,y] \subseteq R \subseteq k[[x,y]]$, R is a subring of $\hat{A}$. So $\omega \in k[x][[y]]$. Conversely, if $\omega \in \hat{A} = k[x][[y]]$ and ω is algebraic over $A = k[x,y]$, then by [1, Theorem 1.1] $\omega \in \tilde{A}$, where $\tilde{A}$ is the henselization of A along Z. So R exists as required. □

If $\omega \in k[[x,y]]$ and ω is algebraic over $k[x,y]$, then the above results show that ω is the image of z under a homomorphism $B = \frac{A[1/\alpha][z]}{(\phi(z))} \longrightarrow k[[x,y]]$. By Proposition 3, $\omega \in k[x][[y]]$ if $A \longrightarrow B$ localizes to an étale neighborhood of the line $y = 0$. Proposition 4 and Corollary 5 give sufficient conditions

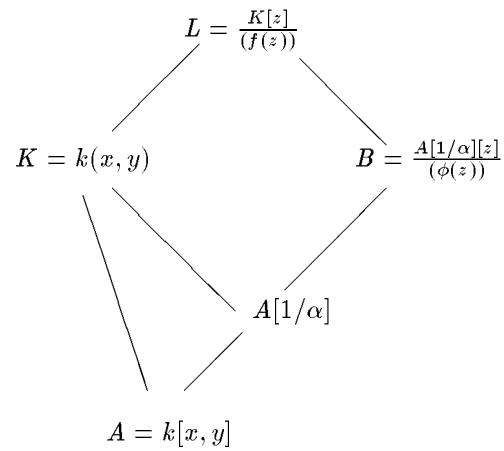

FIGURE 1. Lattice of fields and rings.

identify $\hat{\mathcal{O}}_p$ with $k[[x,y]]$. If $\omega \in k[[x,y]]$, the next lemma gives necessary conditions for ω to be algebraic over $k[x,y]$.

Lemma 1. *Let k be any field and $\omega \in k[[x,y]]$. If ω is algebraic over $k[x,y]$, then there exists an intermediate ring R such that $k[x,y] \subseteq R \subseteq k[[x,y]]$, $\omega \in R$ and $k[x,y] \longrightarrow R$ is an étale neighborhood of $p = Z(x,y)$ in* $\operatorname{Spec} k[x,y]$.

Proof. From [3, p. 37] ω is in $\mathcal{O}_p^h$, hence there is an intermediate ring $\mathcal{O}_p \subseteq R \subseteq \mathcal{O}_p^h$ with the desired properties. □

The context that we really have in mind is the following. The power series ω should be thought of as a function on the xy-plane which defines a surface in xyz-space. We discuss how this situation can occur. Let $A = k[x,y]$, $K = k(x,y)$ and L a finite extension of K obtained by adjoining a root of the irreducible monic polynomial $f(z)$, $\deg f > 1$. So $L = K[z]/(f(z))$. Under what conditions can we view z as a power series in x and y?

For some $\alpha \in A$, there is an irreducible monic polynomial $\phi(z) \in A[1/\alpha][z]$ so that L is the quotient field of $B = \frac{A[1/\alpha][z]}{(\phi(z))}$ and $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A[1/\alpha]$ is a finite morphism between surfaces. That is, if L/K is a finite simple extension, then on some affine open subset $X_1 = \operatorname{Spec} A[1/\alpha]$ of $X = \operatorname{Spec} A$, there is a finite morphism $f\colon Y_1 \longrightarrow X_1$, $Y_1 = \operatorname{Spec} B$, $B = \frac{A[1/\alpha][z]}{(\phi(z))}$. Now assume $\alpha(0,0) \neq 0$ so that $A[1/\alpha] \subseteq k[[x,y]]$ and assume furthermore that there is a power series $\omega \in k[[x,y]]$ such that $z \mapsto \omega$ defines an embedding $B \hookrightarrow k[[x,y]]$. The next lemma, which is a converse of Lemma 1, gives sufficient conditions for the existence of ω.

Lemma 2. *If $f\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is unramified over the origin $p = (0,0)$, then there exists a power series $\omega \in k[[x,y]]$ such that $z \mapsto \omega$ induces an embedding $B \hookrightarrow k[[x,y]]$.*

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When is an Element of $k[[x, y]]$ in $k[x][[y]]$?

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Abstract

Let k be an algebraically closed field and ω a power series in $k[[x, y]]$ that is algebraic over the polynomial ring $k[x, y]$ (i.e. ω satisfies an equation of the form $f(z) = 0$ for some $f(z) \in k[x, y][z]$). Under various conditions on ω and $f(z)$ we give both necessary and sufficient conditions for ω to be a power series in y with coefficients that are polynomials in x.

Throughout k is an algebraically closed field unless explicitly stated otherwise. Suppose ω is a power series in $k[[x, y]]$ which is algebraic over the polynomial ring $k[x, y]$. When is ω a power series in y with coefficients that are polynomials in x? This question was suggested to us by Gabriel Katz. In Proposition 3 below we give necessary and sufficient conditions for this to happen. In Proposition 4 we state sufficient conditions which are more specialized. We conclude with Proposition 6 in which necessary and sufficient conditions are given for the special case where ω satisfies a polynomial of the form $f(z) = z^n - P(x, y)$, $P(x, y)$ a square-free polynomial in $k[x, y]$.

If Z is a closed subvariety of the k-variety X, then an étale neighborhood of Z in X is an étale morphism $U \longrightarrow X$ which is an isomorphism over Z [4]. The henselization of X along Z is the direct limit of all étale neighborhoods of Z.

Let $X = k[x, y]$ and $p = Z(x, y)$ the point in X corresponding to $x = y = 0$. Let $\mathcal{O}_p$ be the local ring on X at p, $\mathcal{O}_p^h$ the henselization of X at p and $\hat{\mathcal{O}}_p$ the completion of $\mathcal{O}_p$ with respect to the m_p-adic topology. We

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