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Zariski Surfaces, Part II
Section 3:
On a Question of Oscar Zariski

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Introduction[1]

We give an example of a surface defined over a field of characteristic 2 with the following properties:

(i) F is an affine surface with equation of the form $F : z^2 = f(x, y)$ in affine 3-space.

(ii) The closure of F in projective space, denoted $\bar{F}$, has a nonsingular mode $\widetilde{F}$ with

$$p_g(\widetilde{F}) = 0 \text{ but } P_2(\widetilde{F}) \geq 1,$$

so that F is nonrational. The example answers in characteristic 2 a question of Professor Zariski which he suggested to P. Blass in 1971. (See (3.0.0)).

Our construction in characteristic 2 follows closely the classical construction of Enriques surfaces as double planes. However, everything had to be done anew in characteristic 2 in order to verify that the classical computations will still go through.

Part 1 is new. On the other hand, our resolution of the singularity at infinity very closely parallels that given in Šafarevič's seminar on surfaces

[1]For Part I of *Zariski Surfaces* see [0].

[4] (pp. 148-165 in the English translation [3]). We acknowledge with pleasure the help of J. S. Milne and M. Hochster in working out the details of this example. Conversation and advice of Professor's Zariski, Mumford, Beauville, Hironka, Bloch and Abhyankar are also gratefully acknowledged.

3.0.0. Zariski's Question (1971). Suppose F is defined by $z^p = f(x,y)$ over an algebraically closed field of characteristic $p > 0$. Let $\widetilde{F}$ be a non-singular projective model of $k(F)$. Is it true that $p_g(\widetilde{F}) = 0$ implies F rational?[2]

3.0.1. Remark (1993) The question is open for $p > 3$. Also it would be interesting to find an example of a hypersurface $F : z^p = f(x,y)$ such that the corresponding Zariski surface $\widetilde{F}$ has $P_2(\widetilde{F}) > 0$ and $p_g(\widetilde{F}) = 0$, where the degree of the defining equation of F is $p+1$ and there are no singularities at infinity.

Part 1. Equation of the surface, singularities in characteristic 2

In this part of section 3 we shall state the equation of the affine surface F and later of its closure in projective space $\bar{F}$. All the non-rational singularities of $\bar{F}$ will be determined. This will be done for a 'general choice'of coefficients entering in the equation of F. It turns out that $\bar{F}$ has only 4 non-rational singularities.

The equation is $F : z^2 = RQ$ over an algebraically closed field k of characteristic 2. Here

$$R(x,y) = xy(x^2 + y^2 + pxy + 1)$$

and

$$Q(x,y) = Ax^4 + A'y^4 + Bxy^3 + B'yx^3 + C(xy)^2 + xy^2 + yx^2 + Ax^2 + A'y^2 + Dxy$$

The curve $Q = 0$ is tangent to the conic at (1,0) and (0,1), and it has a node at (0,0).

We now prove the following:

Proposition 3.0.1 *For a general choice of coefficients p, A, A', B, B', C, D, the affine surface F has only 3 non-rational singularities:* $(0,0,0)$*;* $(0,1,0)$*;* $(0,0,1)$.

Proof. Here we demonstrate that F has no other non-rational singularities. Later (see (3.3.1), (3.5.2), (3.5.3)) it will be shown that these three points are indeed non-rational singularities.

It is easy to show that if (x,y,z) is a non-rational singularity of F then

$$(RQ)_x = \frac{\partial(RQ)}{\partial x} = 0; \tag{1}$$

[2] B. Lang provides a counter-example to (3.0.0) in ([1], [2]) when the characteristic is 3.

$$(RQ)_y = \frac{\partial(RQ)}{\partial y} = 0; \tag{2}$$

$$(RQ)_{xy} = \frac{\partial^2(RQ)}{\partial x \partial y} = 0; \tag{3}$$

(See (3A.3.0) in the appendix to this section.)

These are equivalent to

$$(1)' \qquad R_x Q + Q_x R = 0;$$

$$(2)' \qquad R_y Q + Q_y R = 0;$$

$$(3)' \qquad R_{xy} Q + Q_{xy} R + R_x Q_y + Q_x R_y = 0.$$

Let $\Delta = \det \begin{vmatrix} R_x & Q_x \\ R_y & Q_y \end{vmatrix} = R_x Q_y + Q_x R_y$. We will show that $\Delta = 0$ at a point satisfying $(1)'$, $(2)'$, $(3)'$. If Δ were not 0 at such a point, then $(1)'$ and $(2)'$ imply $R = Q = 0$; but then $(3)'$ implies $\Delta = 0$; a contradiction. Thus

$$\Delta = 0 \text{ as a consequence of } (1)', (2)', (3)'. \tag{4}$$

To evaluate Δ observe that

$$R_x = x^2 y + y^3 + y = y(x^2 + y^2 + 1) = y(x + y + 1)^2, \tag{5}$$
$$R_y = x(x^2 + y^2 + 1) = x(x + y + 1)^2, \tag{6}$$
$$Q_x = By^3 + B'yx^2 + y^2 + Dy = y(By^2 + B'x^2 + y + D), \tag{7}$$
$$Q_y = B'x^3 + Bxy^2 + x^2 + Dx = x(B'x^2 + By^2 + x + D), \tag{8}$$

so that

$$Q_{xy} = By^2 + B'x^2 + D, \tag{9}$$

and therefore

$$\begin{aligned} \Delta &= (x + y + 1)^2 (yQ_y + xQ_x) \\ &= (x + y + 1)^2 (xy)(By^2 + B'x^2 + y + D + B'x^2 + By^2 + x + D) \\ &= (x + y + 1)^2 (xy)(x + y) \end{aligned}$$

Thus if a singularity satisfies $(1)'$, $(2)'$, $(3)'$ there are four possible cases:

(a) $x = 0$, (b) $y = 0$, (c) $x + y = 0$, (d) $x + y + 1 = 0$.

Case (a) Since $x = 0, R = Q_y = 0$, so that $(1)'$ and $(2)'$ reduce to $R_x Q = R_y Q = 0$. If $Q = 0$, then $x = 0$ implies

$$A'y^4 + A'y^2 = A'(y^2 + y)^2 = 0,$$

so that if we assume $A' \neq 0$, then $y = 0$ or $y = 1$, and we obtain only $(0, 0, 0)$ or $(0, 1, 0)$. If $Q \neq 0$, then $R = R_x = R_y = 0$ and again we get only $(0, 0, 0)$ or $(0, 1, 0)$.

Case (b) Since F is symmetric in x and y, if we assume $A \neq 0$, then from case (a) we obtain only (0,0,0) or (1,0,0).

Case (c) Here $x = y$. Then from (5)-(9) we have $R_x = x, R_y = x, Q_x = x[(B + B')x^2 + x + D] = Q_y, R_{xy} = 1$ and $Q_{xy} = (B + B')x^2 + D$. Also, Q and R become $Q = (A + A' + B + B' + C)x^4 + (A + A' + D)x^2$ and $R = x^2(px^2 + 1)$.

From this it follows that

$$R_xQ + Q_xR = x[(A + A' + B + B' + C)x^4 + (A + A' + D)x^2] \\ + [(B + B')x^2 + x + D]x^3(px^2 + 1) = 0$$

and

$$R_{xy}Q + Q_{xy}R = (A + A' + B + B' + C)x^4 + (A + A' + D)x^2 \\ + [(B + B')x^2 + D]x^2(px^2 + 1) = 0.$$

If $x = 0$, then $y = 0$, so there is no new solution. If $x \neq 0$, divide $R_xQ + Q_xR$ by x^3 to get

$$(A + A' + B + B' + C)x + (A + A' + D) + [(B + B')x^2 + x + D](px^2 + 1) = 0$$

and divide $R_{xy} + Q_{xy}R$ by x^2 to get

$$(A + A' + B + B' + C)x^2 + (A + A' + D) + [(B + B')x^2 + D](px^2 + 1) = 0.$$

Subtract these two equations to obtain $x(px^2 + 1) = 0$ and hence $px^2 = 1$ and $(A+A'+B+B'+C)x^2+(A+A'+D) = 0$. Thus $(A+A'+B+B'+C) = p(A + A' + D)$. This last equality will not hold for a general choice of coefficients, so that for such a general choice the only solution of $(1)'$, $(2)'$, $(3)'$ with $x = y$ is $x = y = z = 0$.

Case (d) Here $y = x+1$. $R_x = R_{xy} = \Delta = 0$, so that $(1)'$ and $(3)'$ become $Q_xR = 0$ and $Q_{xy}R = 0$, respectively.
If $R = 0$, then $px^2(x + 1)^2 = 0$. If we assume $P \neq 0$, then $x = 0$ or $x = 1$. $x = 0$ gives (0,1,0) and $x = 1$ gives (1,0,0).

If $R \neq 0$, then $Q_x = Q_xy = 0$; which by (7) and (8) yields $y^2 = 0$ and hence $y = 0$. But $y = 0$ implies $x = 1$ and we only obtain the point (1,0,0).

To determine the singularities at infinity, homogenize the equation of F:

$$\bar{F} : z_0^6 z^2 = xy(x^2 + y^2 + pxy + z_0^2)Q(x, y, z_0)$$

where

$$Q(x, y, z_0) = Ax^4 + A'y^4 + Bxy^3 + B'yx^3 + C(xy)^2 + z_0(xy^2 + yx^2) + z_0^2(Ax^2 + A'y^2 + Dxy).$$

Let $F_8(x, y, z_0)$ denote the right-hand side of the equation of $\bar{F}$ which is a homogeneous form of degree 8. Also,

$$z_0^6 z^2 = f_8(x, y) + z_0 F_7(x, y, z_0)$$

where

$$f_8(x, y) = xy(x^2 + y^2 + pxy)(Ax^4 + A'y^4 + Bxy^3 + B'x^3y + C(xy)^2)$$

and F_7 is a homogeneous form of degree 7.

A singularity at infinity has the form $(x, y, z, 0)$ and it must satisfy

$$\frac{\partial f_8}{\partial x}(x, y) = \frac{\partial f_8}{\partial y}(x, y) = \frac{\partial F_8}{\partial z_0}(x, y, z_0) = f_8(x, y) = 0$$

For a general choice of p, A, A', B, B' and C, f_8 has distinct linear factors; thus $f_8 = (f_8)_x = (f_8)_y = 0$ forces $x = y = 0$; so we obtain exactly one singular point at infinity with $x = y = z_0$ and $z = 1$. The local equation is

$$F_0 : z_0^6 = f_8(x, y) + z_0 F_7(x, y, z_0) = F_8(x, y, z_0)$$

Remark 3.0.2**.** For a general choice of coefficients, $Sing(\bar{F})$ consists of a finite number of isolated singularities.

Proof. There is only one singular point on $\bar{F} - F$. By (3.0.1) F has only three singularities satisfying (1), (2) and (3). All of the other singularities are isolated by (3A.3.0). Thus all singularities of F are isolated. Since $Sing(\bar{F})$ is closed in the Zariski topology of $\bar{F}, Sing(\bar{F})$ is finite.

Remark 3.0.3**.** For a general choice of coefficients, $\bar{F}$ is a normal surface. This follows from (3.0.2) since a projective hypersurface of dimension 2 with isolated singularities is normal.

Part 2

In this part the singularities of F are resolved, and adjoints and bi-adjoints are determined locally at each of the three non-rational singular points (Corollaries (3.3.1), (3.3.2), (3.5.1), (3.5.2), (3.5.3), (3.5.4), (3.5.5), (3.5.6)).

Lemma 3.1.0. *Suppose that k is an algebraically closed field of characteristic 2 and*

$$w^2 = f_4(u,v) + f_5(u,v) + \cdots + f_n(u,v), \tag{1}$$

with $n \geq 5$ is an equation where $f_i(u,v) \in k[u,v]$ is homogeneous of degree i, $4 \leq i \leq n$. Suppose that the singularity at the point $u = v = w = 0$ is isolated. Suppose also that the following condition, referred to later as (3.1.1) *is satisfied:*

(3.1.1) *Neither the system* $\dfrac{\partial f_4}{\partial u}(u,1) = f_5(u,1) = 0$ *nor the system* $\dfrac{\partial f_4}{\partial v}(1,v) = f_5(1,v) = 0$ *has a solution.*

Then (a) equation (1) defines an irreducible hypersurface

$$S = \operatorname{Spec} \frac{k[u,v,w]}{(w^2 - f_4 - f_5 - \cdots - f_n)} = \operatorname{Spec}\ k[S]$$

and

(b) the singularity at (0,0,0) is resolved by a quadratic transformation followed by blowing up a double line which is the exceptional locus.

Proof of (a). To show that S is irreducible, it is enough to see that the right hand side of (1) is not a square since $f_5 \neq 0$.

Proof of (b). First we perform a quadratic transformation. We blow up the ideal (u,v,w). Since w is integrally dependent on (u,v), therefore by (3A.2.0) the following two affine schemes cover QS, the blown up variety:

$$\operatorname{Spec} R' = \operatorname{Spec}\ k\left[u, \frac{v}{u}, \frac{w}{u}\right] = \operatorname{Spec}\ k[u', v', w']$$

and

$$\operatorname{Spec}\ R'' = \operatorname{Spec}\left[\frac{u}{v}, v, \frac{w}{v}\right] = \operatorname{Spec}\ k[u'', v'', w''].$$

These schemes can be identified with hypersurfaces having equations

$$(w')^2 = (u')^2 f_4(1,v') + \cdots + (u')^{n-2} f_n(1,v'), \tag{2}$$
$$(w'')^2 f_4(u'',1) + \cdots + (v'')^{n-2} f_n(u'',1), \tag{3}$$

respectively. The glueing together is by the isomorphism of open sets $(\operatorname{Spec} R')_{(v')} = (\operatorname{Spec} R'')_{(u'')}$ that is given by $R'\left[\frac{u}{v}\right] = R''\left[\frac{v}{u}\right]$.

Next we blow up the 'double line', i.e., the coherent sheaf of ideals given by $(w', u') = (w/u, u)$ on Spec R' and $(w'', v'') = (w/v, v)$ on Spec R''. Since, in the ring of intersection $k[u, v/u, w/u, u/v], (w/u, u) = (w/v, v)$, therefore the sheaf of ideals is well defined on QS. Blowing it up gives a scheme covered by two affines since w' is integrally dependent on the ideal (u') and w'' on the ideal (v'').

The new scheme $\widetilde{S}$ is thus covered by the spectra of the following two rings:

$$R_1 = k\left[u', v', \frac{w'}{u'}\right] = k\left[u, \frac{v}{u}, \frac{w}{u^2}\right] = k[u_1, v_1, w_1]$$

and

$$R_2 = k\left[u'', v'', \frac{w''}{v''}\right] = k\left[\frac{u}{v}, v, \frac{w}{v^2}\right] = k[u_2, v_2, w_2].$$

Again, by (3A.2.0), R_1 and R_2 are coordinate rings of the following hypersurfaces:

$$\widetilde{S}_1 : w_1^2 = f_4(1, v_1) + u_1 f_5(1, v_1) + \cdots + u_1^{n-4} f_n(1, v_1) \qquad \text{(a)}$$

$$\widetilde{S}_2 : w_2^2 = f_4(u_2, 1) + v_2 f_5(u_2, 1) + \cdots + v_2^{n-4} f_n(u_2, 1). \qquad \text{(b)}$$

The glueing together is given by $\mathrm{Spec}(R_1) \cap \mathrm{Spec}(R_2) = \mathrm{Spec}\ R_1\left[\frac{1}{v_1}\right] = \mathrm{Spec}\ R_2\left[\frac{1}{u_2}\right] = \mathrm{Spec}(\text{compositum } R_1R_2 \text{in } k(S))$. We note that $k(S) = k(u, v, w)$. $\widetilde{S}$ has a natural projection down to S. Call it

$$\pi : \widetilde{S} \to S.$$

That projection restricted to Spec $R_1 = \widetilde{S}_1$ and Spec $R_2 = \widetilde{S}_2$ corresponds to the natural inclusions

$$k[u, v, w] \hookrightarrow R_1 = k\left[u, \frac{v}{u}, \frac{w}{u^2}\right] = k[u_1, v_1, w_1]$$

$$k[u, v, w] \hookrightarrow R_2 = k\left[v, \frac{u}{v}, \frac{w}{v^2}\right] = k[u_2, v_2, w_2].$$

To find the fibre over the point (0,0,0) on S we observe that in R_1 the ideal (u, v, w) is equal to $(u) = (u_1)$; and in R_2 (u, v, w) is equal to $(v) = (v_2)$. Thus the exceptional fibre is the curve $u_1 = 0$ on $\widetilde{S}_1$ glued together with the curve $v_2 = 0$ on $\widetilde{S}_2$. Equations (a) and (b) together with condition (3.1.1) and the Jacobian criterion suffice to show that all points of the exceptional fibre on $\widetilde{S}$ are simple for $\widetilde{S}$. Hence the singularity is resolved by

$$\pi : \widetilde{S} \to S. \qquad \square$$

Lemma 3.2.0 *Let* $S: w^2 = f_4(u,v) + f_5(u,v) + \cdots + f_n(u,v) = f(u,v)$ *be as in* (3.1.0), *satisfying* (3.1.1). *In addition, we assume condition* (3.2.1), *namely, that* $f_4(1,u)$ *and* $f_4(v,1)$ *are not perfect squares (characteristic 2).*

Then a monomial $u^a v^b w^c$ *is adjoint at the isolated singular point* $u = v = w = 0$ *if and only if* $a + b + 2c \geq 1$, *and it is biadjoint if and only if* $a + b + 2c \geq 2$.

Proof. We continue with the notations of (3.1.0): $\widetilde{S} \xrightarrow{\pi} S$ is the resolution, S is covered by $\widetilde{S}_1$ and $\widetilde{S}_2$, the two affine hypersurfaces with equations

$$\widetilde{S}_1 : w_1^2 = f_4(1,v_1) + u_1 f_5(1,v_1) + \cdots + u_1^{n-4} f_n(1,v_1) = g_1(u_1,v_1)$$

and

$$\widetilde{S}_2 : w_2^2 = f_4(u_2,1) + v_2 f_5(u_2,1) + \cdots + v_2^{n-4} f_n(u_2,1) = g_2(u_2,v_2).$$

Denote the restriction of π to $\widetilde{S}_i$ by π_i. π_1, π_2 are dual to the ring inclusions π_1^*, π_2^* given by

$$k[u,v,w] \overset{\pi_1^*}{\hookrightarrow} k\left[u, \frac{v}{u}, \frac{w}{u^2}\right] = k[u_1, v_1, w_1],$$

$$k[u,v,w] \overset{\pi_2^*}{\hookrightarrow} k\left[\frac{u}{v}, v, \frac{w}{v^2}\right] = k[u_2, v_2, w_2].$$

To determine adjoints we compute the pullback of the canonical generating differential σ_s to $\widetilde{S}_1$ and $\widetilde{S}_2$.

$$\sigma_s = \frac{du\ dw}{\frac{\partial f}{\partial v}} = \frac{dv\ dw}{\frac{\partial f}{\partial u}}.$$

Since $\pi_1^*(u) = u_1, \pi_1^*(v) = u_1 v_1, \pi_1^*(w) = u_1^2 w_1$, and $\pi_2^*(u) = u_2 v_2, \pi_2^*(v) = v_2, \pi_2^*(w) = v_2^2 w_2$, we have

$$\pi_1^{-1}(\sigma_s) = \frac{u_1^2\ du_1 dw_1}{\frac{\partial f}{\partial v}(u_1, u_1 v_1)} \text{ and } \pi_2^{-1}(\sigma_s) = \frac{v_2^2 dv_2 dw_2}{\frac{\partial f}{\partial u}(v_2 u_2, v_2)}.$$

But $g_1(u_1,v_1) = \dfrac{1}{u_1^4} f(u_1, u_1 v_1)$ and $g_2(u_2,v_2) = \dfrac{1}{v_2^4} f(u_2 v_2, v_2)$, thus by the chain rule

$$\frac{\partial g_1}{\partial v_1} = \frac{1}{u_1^4}\left(\frac{\partial f}{\partial v}\right)(u_1, u_1 v_1)\cdot u_1 = \frac{1}{u_1^3}\left(\frac{\partial f}{\partial v}\right)(u_1, u_1 v_1).$$

Similarly $\dfrac{\partial g_2}{\partial u_2}(u_2,v_2) = \dfrac{1}{v_2^3}\dfrac{\partial f}{\partial u}(u_2, u_2 v_2)$. Therefore $\pi_1^{-1}(\sigma_s) = \dfrac{1}{u_1} \dfrac{du_1 dw_1}{\dfrac{\partial g_1}{\partial v_1}(u_1,v_1)} = \dfrac{1}{u_1}\sigma_{\widetilde{S}_1}$ and $\pi_2^{-1}(\sigma_s) = \dfrac{1}{v_2}\sigma_{\widetilde{S}_2}$ where $\sigma_{\widetilde{S}_1}, \sigma_{\widetilde{S}_2}$ are canonical generating differentials on $\widetilde{S}_1$, $\widetilde{S}_2$ regular along the exceptional fibre.

The exceptional fibre of the map $\pi : \widetilde{S} \to S$ is glued together from the curve $u_1 = 0$ on $\widetilde{S}_1$ and $v_2 = 0$ on $\widetilde{S}_2$.

These curves on $\widetilde{S}_1$ and $\widetilde{S}_2$ define discrete valuations ρ_1 of $k(\widetilde{S}_1)$ and ρ_2 of $k(\widetilde{S}_2)$ such that for all nonnegative integers α, β, γ

$$\rho_1(u_1^\alpha v_1^\beta w_1^\gamma) = \alpha \qquad \text{and} \qquad \rho_2(u_2^\alpha v_2^\beta w_2^\gamma) = \beta.$$

(Here we need condition (3.2.1). See (3A.1.0) in the appendix.) Under the canonical identification of $k(\widetilde{S}_1) = k(\widetilde{S}_2) = k(S) = k(u, v, w), \rho_1 = \rho_2$. (The valuation measures the order of pole or zero along the exceptional fibre.) We write $\rho = \rho_1 = \rho_2$.

Since $\rho(\sigma_{\widetilde{S}_1}) = \rho(\sigma_{\widetilde{S}_2}) = 0$ (see [0], (2.34.0)), we obtain $\rho(\pi_1^{-1}(\sigma_s)) = \rho(\pi_2^{-1}(\sigma_s)) = -1$. Thus a monomial $u^a v^b w^c$ is adjoint at (0,0,0) if and only if it pulls back on $\widetilde{S}_1$ and $\widetilde{S}_2$ to a function of value≥ 1. Since $\pi_1^{-1}(u^a v^b w^c) = u_1^{a+b+2c} v_1^b w_1^c$ and $\pi_2^{-1}(u^a v^b w^c) = v_2^{a+b+2c} u_2^a w_2^2$ we see that $u^a v^b w^c$ is adjoint if and only if $a + b + 2c \geq 1$. Similarly by considering the pullback of $w_s^{\otimes 2}$ we conclude that $u^a v^b w^c$ is biadjoint if and only if $a + b + 2c \geq 2$ (see (3A.1.1)). □

Corollary 3.2.1 *$p(u, v, w) \in k[u, v, w] = k[S]$ is adjoint at $(0, 0, 0)$ if and only if $p(0, 0, 0) = 0$.*

Corollary 3.2.2 *$p(u, v, w)$ is biadjoint if and only if p is a linear combination of monomials $u^a v^b w^c$ satisfying $a + b + 2c \geq 2$.*

Proof. We need only prove the 'only if' part. The only monomials not satisfying $a + b + 2c \geq 2$ are u, v and nonzero elements of k. Therefore we can write $p(u, v, w) = p_1(u, v, w) + c_1 u + c_2 v + c_3$ where p_1 is biadjoint and $c_1, c_2, c_3 \in k$.

It is enough to show that p biadjoint implies $c_1 = c_2 = c_3 = 0$. If p is biadjoint, then p is adjoint, so $c_3 = 0$ by (3.2.1). Thus $c_1 u + c_2 v$ is biadjoint. Since the pullback $\pi_1^{-1}(c_1 u + c_2 v) = u_1(c_1 + c_2 v_1)$, we must have $\rho(u_1(c_1 + c_2 v_1)) \geq 2$; which implies $\rho(c_1 + c_2 v_1) \geq 1$. But if $(c_1, c_2) \neq (0, 0)$, then by lemma (3A.1.0), part (iii), $\rho(c_1 + c_2 v_1) = 0$. Therefore $c_1 = c_2 = 0$. □

Proposition 3.3.0 *Take the equation of F,*

$$z^2 = xy(x^2 + y^2 + pxy + 1)Q,$$

where $Q = Ax^4 + A'y^4 + Bxy^3 + B'yx^3 + C(xy)^2 + xy^2 + yx^2 + Ax^2 + A'y^2 + Dxy$. Then for a general choice of coefficients the origin $(0, 0, 0)$ is an isolated singularity that satisfies conditions (3.1.1) and (3.2.1).

Proof. By (3.0.2) we may assume that $(0, 0, 0)$ is an isolated singularity. We have $z^2 = xy(Ax^2 + A'y^2 + Dxy) + xy(xy^2 + yx^2)$ + higher order terms.

We use the notations of lemma (3.1.0) with $u = x, v = y, w = z$. Then

$$f_4(x,1) = x(Ax^2 + A' + Dx),$$
$$f_5(x,1) = x^2(x+1),$$
$$\frac{\partial f_4}{\partial x}(x,1) = Ax^2 + A'.$$

If $(A + A')A' \neq 0$, then $f_5(x,1)$ and $\frac{\partial f_4}{\partial x}(x,1)$ have no common roots. Similarly, if $(A + A')A \neq 0$ then $f_5(1,y)$ and $\frac{\partial f_4}{\partial y}(1,y)$ have no common roots. If $AA' \neq 0, f_4(x,1)$ and $f_4(1,y)$ are not perfect squares. Therefore, for the general choice of coefficients $(A+A')AA' \neq 0$ conditions (3.1.1) and (3.2.1) are satisfied. □ We restate (3.2.1) in terms of adjoint surfaces (see (2.16.1) for the definition of an adjoint surface).

Corollary 3.3.1 *Adjoint surfaces to our surface F pass through the singular point $(0,0,0)$. Conversely any surface that passes through $(0,0,0)$ is adjoint.*

Corollary 3.3.2 *xy is biadjoint at $(0,0,0)$.*

Proof. Follows from (3.2.2) and (3.3.0). □

Proposition 3.4.0 *The singularity of F at $(0,1,0)$ is resolved as follows. After a single quadratic transform one obtains in the exceptional locus only one isolated singular point. It is a quadruple point satisfying (3.1.1) and (3.2.1).*

This is true for a general choice of parameters in the equation of F. The same fact is true for the singularity of F at (1,0,0).

Proof. The last statement follows by symmetry of the equation of F in terms of x and y and by interchanging x and y, A and A', B and B', etc. in the below proof. For a general choice of coefficients $(0,1,0)$ is an isolated singularity. To get the local equation of F replace y by $\bar{y}+1$ to get $z^2 = x(\bar{y}+1)(x^2+\bar{y}^2+px\bar{y}+px)\bar{Q}$ where

$$\bar{Q} = Q(x,\bar{y}) + x(B+D+1) + x^2(C+1) + B'x^3 + Bx(\bar{y}^2+\bar{y}).$$

Blow up the ideal $(x,\bar{y},z)$ to obtain a new variety QF. Since z is integrally dependent on $(x,\bar{y}), QF$ is (by (3A.2.0)) covered by two affines E_1, E_2. The first affine chart E_1 is given by (see (3A.2.0)) $E_1 = \text{Spec } R_1$ where

$$R_1 = k\left[x, \frac{\bar{y}}{x}, \frac{z}{x}\right] = k[x', \bar{y}', z'].$$

E_1 is then a hypersurface with equation

$$z'^2 = x'(\bar{y}'x'+1)(x'+\bar{y}'^2x'+px'\bar{y}'+p)(B+D+1) + x'\widetilde{Q}(x',y'),$$

where $\widetilde{Q}(x', y')$ is some polynomial. Along the exceptional fibre given by $x' = 0, \partial/\partial x'$ is easily computed and equals $p(B + D + 1)$. This is $\neq 0$ if we assume $p \neq 0$ and $B + D + 1 \neq 0$. And under these assumptions there are no singularities along the exceptional locus in this chart.

The second affine chart E_2 is given, again by (3A.2.0), by $E_2 =$ Spec R_2 where

$$R_2 = k\left[\frac{x}{\bar{y}}, \bar{y}, \frac{z}{\bar{y}}\right] = k[x'', y'', z''].$$

Again E_2 is a hypersurface, which is given by

$$E_2 : z''^2 = x"\bar{y}''(\bar{y}'' + 1)(x''^2\bar{y}'' + \bar{y}'' + px''\bar{y}'' + px'')[(B + D + 1)x'' \\ + A'\bar{y}'' + (B + D)x''\bar{y}'' + \bar{y}''Q''(x'', \bar{y}'')],$$

where $Q''(x'', \bar{y}'')$ contains only monomials of degree two or higher.

Along the exceptional locus $\bar{y}'' = 0$ and therefore $\partial/\partial\bar{y}''$ is given by $x''px''(X + D + 1)x''$. Thus if we assume $p(B + D + 1) \neq 0, x'' = \bar{y}'' = z'' = 0$ is the only singular point. For a generic choice of coefficients, the singular point will be isolated since the original singularity is isolated. It is quadruple, so it is only left to check conditions (3.1.1) and (3.2.1). Hereafter we shall drop the primes and bars.

We have $z^2 = f_4(x, y) + f_5(x, y) +$ higher order terms, where $f_4(x, y) = xy(px + y)(\beta x + A'y)$ with $\beta = B + D + 1$. Also $f_5(x, y) = xy^2[(px + y)(\beta x + A'y) + px(\beta x + A'y) + (px + y)(B + D)x]$. Consequently,

$$f_4(x, 1) = x(px + 1)(\beta x + A')$$

and

$$f_4(1, y) = y(p + y)(\beta + A'y).$$

The partial derivatives give

$$\frac{\partial f_4}{\partial y}(1, y) = A'y^2 + p\beta \tag{1}$$

and

$$\frac{\partial f_4}{\partial x}(x, 1) = p\beta x^2 + A' \tag{2}$$

Also

$$(1)' \quad \begin{aligned} f_5(1, y) &= y^2\left[(p + y)(\beta + A'y) + p(\beta + A'y) + (B + D)(p + y)\right] \\ &= y^2\left[A'y^2 + (pA' + \beta + pA\prime + B + D)y + p(B + D)\right] \\ &= y^2\left[A'y^2 + (\beta + \beta + 1)y + p(B + D)\right] \\ &= y^2\left[A'y^2 + y + p(\beta + 1)\right] \end{aligned}$$

Similarly $f_5(x, 1)$ simplifies to

$$(2)' \quad f_5(x, 1) = x(x^2p(\beta + 1) + x + A').$$

We have to show that for a general choice of parameters, (1), $(1)'$ have no common root, and the same for (2) and $(2)'$.

If $\dfrac{\partial f_4}{\partial y}(1,y) = f_5(1,y) = 0$, then $y^2(y+p) = 0$; which implies $p\beta = 0$ or $A'p = \beta$, but this is not true for a general choice of coefficients.

If $\dfrac{\partial f_4}{\partial x}(x,1) = f_5(x,1) = 0$, then $x^2(xp+1) = 0$; which implies $A' = 0$ or $\beta = A'p$, and again this is not the case for a general choice of parameters.

To check condition (3.2.1) observe that in the above notation $f_4(1,y) = y(y+p)(\beta + A'y)$ is not a perfect square if $A' \neq 0$ in characteristic 2. Also $f_4(x,1) = x(px+1)(\beta x + A')$ is not a square if $p\beta \neq 0$. □

Proposition 3.5.0 *Let $\bar{y} = y+1$ and let F be our surface. A monomial $x^a \bar{y}^b z^c$ is adjoint at $(0,1,0)$ if and only if $b + 2a + 3c \geq 1$ and biadjoint if and only if $b + 2a + 3c \geq 2$.*

Proof. From the resolution (3.4.0) we know that if we write F in terms of x and $\bar{y}$, then we have

$$F : z^2 = f_3(x,\bar{y}) + f_4(x,\bar{y}) + \cdots + f_8(x,\bar{y}) = f(x,\bar{y}).$$

Also if we blow up $(x,\bar{y},z)$ we obtain QF covered by two affine hypersurfaces:

$$E_1 = \text{Spec } k\left[x, \frac{\bar{y}}{x}, \frac{z}{x}\right] = \text{Spec } k[x,\bar{y}',z']$$

where

$$E_1 : z'^2 = \frac{1}{x'^2} f(x', \bar{y}'x') = g_1(x',\bar{y}')$$

and

$$E_2 = \text{Spec } k\left[\frac{x}{\bar{y}}, \bar{y}, \frac{z}{\bar{y}}\right] = \text{Spec } k[x'',\bar{y}'',z'']$$

with equation

$$E_2 : z''^2 = \frac{1}{\bar{y}''^2} f(x''\bar{y}'', \bar{y}'') = g_2(x'',\bar{y}'').$$

E_1 is non-singular along the exceptional fibre and E_2 has only one singular point $x'' = \bar{y}'' = z'' = 0$ on the exceptonal fibre. That singular point is an isolated singularity of E_2 satisfying (3.1.1) and (3.2.1).

By (3.2.0) a function

$$x''^{\alpha}\bar{y}''^{\beta}z''^{\gamma} \text{ is } \begin{cases} \text{adjoint if and only if } \alpha+\beta+2\gamma \geq 1, \\ \text{biadjoint if and only if } \alpha+\beta+2\gamma \geq 2. \end{cases}$$

We first compute the pullback of the canonical generating differential on F to E_1. The projection of E_1 to F is given by

$$k[x,\bar{y},z] \hookrightarrow k\left[x, \frac{\bar{y}}{x}, \frac{z}{x}\right] = k[x',\bar{y}',z']$$

so that $x \to x', \bar{y} \to \bar{y}'x', z \to z'x'$. We write

$$\sigma_F = \frac{dx\ dz}{\frac{\partial f}{\partial \bar{y}}}.$$

The pullback to E_1 is

$$\frac{x' dx' dz'}{\left(\frac{\partial f}{\partial \bar{y}}\right)(x', \bar{y}'x')}$$

but

$$\frac{\partial g_1}{\partial \bar{y}'}(x', \bar{y}') = \frac{1}{x'}\left(\frac{\partial f}{\partial \bar{y}}\right)(x', \bar{y}'x').$$

Therefore the pullback of σ_F to E_1 is

$$\frac{dx' dz'}{\left(\frac{\partial g_1}{\partial \bar{y}'}\right)(x', \bar{y}')}$$

which is a canonical generating differential on E_1, σ_{E_1}, regular along the exceptional fibre which is nonsingular.

Similarly σ_F pulls back to

$$\frac{d\bar{y}'' dz''}{\left(\frac{\partial g_2}{\partial \bar{y}''}\right)(x'', \bar{y}'')}$$

the generating differential σ_{E_2} on E_2.

Because of the above computations and since E_1 is already nonsingular it is clear that a function $p(x, \bar{y}', z) \in k[x, \bar{y}, z]$ is adjoint at (0,0,0) if and only if its pullback to E_2 is locally adjoint at $x'' = \bar{y}'' = z'' = 0$.

Taking a monomial $x^a \bar{y}^b z^c$ its pullback is $x''^a (\bar{y}'')^{a+b+c} z''^c$. This is adjoint by (3.2.0) if and only if $b + 2a + 3c \geq 1$ and is biadjoint if and only if $b + 2a + 3c \geq 2$. □

Corollary 3.5.1 *Let $p \in k[F] = k[x, y, z]$, let $\bar{y} = y + 1$ and write $p = p(x, \bar{y}, z)$; then $p(x, \bar{y}, z)$ is adjoint at $x = \bar{y} = z = 0$ if and only if $p(0,0,0) = 0$.*

Corollary 3.5.2 *Adjoint surfaces to our surface F pass through $(1,0,0)$. Conversely, any surface passing through $(1,0,0)$ is adjoint to F at that point.*

Corollary 3.5.3 *Adjoint surfaces pass through (0,1,0) and any surface passing through (0,1,0) is adjoint to F at that point.*

Proof. Symmetry in x, y. □

Corollary 3.5.4 *Using notations of* (3.5.1), $p(x, \bar{y}, z) \in k[x, \bar{y}, z]$ *is biadjoint at* $(0, 1, 0)$ *if and only if* p *is a linear combinatin of monomials* $x^a \bar{y}^b z^c$ *satisfying* $2a + b + 3c \geq 2$.

Proof. The only monomials not satisfying the above are the nonzero constants and $\bar{y}$. It is enough to show that $c_1 + c_2\bar{y}$ is biadjoint if and only if $c_1 = c_2 = 0$. By (3.5.1) $c_1 = 0$ since a biadjoint is adjoint. Pulling back $c_2\bar{y}$ to E_2, $c_2\bar{y}$ is biadjoint at $x'' = \bar{y}'' = z'' = 0$, but this contradicts (3.2.0) unless $c_2 = 0$. □

Corollary 3.5.5 $xy = x\bar{y} + x$ *is biadjoint at* $(0, 1, 0)$.

Corollary 3.5.6 xy *is biadjoint at* $(1, 0, 0)$.

Proof. By symmetry

Remark 3.5.7. (3.3.1), (3.3.2) and (3.5.1)-(3.5.6) are true for a general choice of the coefficients of F.
(To be continued in the next issue.)

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