3. F. Cossec and I. Dolgachev, *Enriques Surfaces I*, vol. 76 of Progress in Mathematics, Birkhäuser, Boston/Basel, 1989.
4. T. J. Ford, *The Brauer group and ramified double covers of surfaces*, Comm. Algebra, **20** (1992), pp. 3793–3803.

(h) How about if we assume S is a complete intersection?

(3) This conjecture concerns the surface X defined by the affine equation $z^2 = C_i C_j$ where C_i is a sufficiently general curve of degree i and $i + j = 2m$ for some $m \geq 3$. Furthermore we assume X is not a ruled surface. The second Betti number of X is given by the formula (see [3, p. 15]) $b_2(X) = 2(2m^2 - 3m + 2)$.

(a) Suppose $i + j = 6$. Then X is K3, $b_2(X) = 22$. If $z^2 = C_6$, then X has minimal Picard number 1. The conjecture is that the Picard group of X is generated by the exceptional fibers lying over the singular points of the curve $C_i C_j$, and a hyperplane section. There are 3 cases:

(i) $z^2 = C_1 C_5$. Conjecture: $\operatorname{Pic} X = \mathbb{Z}^6$.

(ii) $z^2 = C_2 C_4$. Conjecture: $\operatorname{Pic} X = \mathbb{Z}^9$.

(iii) $z^2 = C_3 D_3$. Conjecture: $\operatorname{Pic} X = \mathbb{Z}^{10}$.

(b) Generally, $z^2 = C_i C_j$, $i + j = 2m$. Conjecture: $\operatorname{Pic} X = \mathbb{Z}^{ij+1}$.

(4) Consider the surface $z^2 = C_i C_j$ as in 5 above, $i + j = 2m$. Dehomogenize: $z^2 = C_i(x, y, 1) C_j(x, y, 1)$. Conjecture: this affine surface has $\mathrm{Cl}(X) = \mathbb{Z}/2$. Is this also true in characteristic 2?

(5) Consider the surface $z^2 = xy + h_3 + h_4 + h_5 + h_6$, where h_i is a general form of degree i. Conjecture: this surface is generically factorial, hence any node has trivial class group.

(6) Let C_6 be a generic curve with a node at (0,0,1) and X the nonsingular complete minimal model for $z^2 = C_6$. Conjecture: $\operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}$ and the affine surface $z^2 = C_6$ is factorial.

(7) Consider the following examples and try to explain:

(a) Let X be a sufficiently general quintic with a triple point at 0: $x^3 + y^3 + z^3 + f_4 + f_5 = 0$. Then 1 blow up $\tilde{X} \to X$ resolves the singularity and the exceptional fiber is an elliptic curve E. Then

$$0 \to \mathrm{B}(\tilde{X}) \to \mathrm{B}(\tilde{X} - E) \to H^1(E, \mathbb{Q}/\mathbb{Z}) \to H^3(\tilde{X}, \mathbb{Q}/\mathbb{Z})$$

is exact. Claim: $p_g = p_a = 3$ for $\tilde{X}$, which implies $b_1 = 0$, hence $H^3 = 0$. So there exist division algebras over K which ramify on the singular point of X.

(b) If X is a cone over E, the above doesn't happen: $\mathrm{B}(\tilde{X}) = \mathrm{B}(\tilde{X} - E)$.

(8) Let X be a normal surface with an isolated singularity p. Assume $\mathrm{Cl}(\mathcal{O}_p) = 0$. Let $X_1 \to X$ be a blow up of p followed by a normalization. Are all of the new local rings still factorial?

References

1. J. Blass, P. Blass, and T. J. Ford, *On a remark of Grothendieck*, Comm. Algebra, **18** (1990), pp. 3685–3687.
2. P. Blass and R. Hoobler, *Picard and Brauer groups of Zariski surfaces*, Proc. Amer. Math. Soc., **97** (1986), pp. 379–383.

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Open Questions

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Abstract

The following open questions are submitted to the Electronic Scottish Book.

(1) What about Hoobler's computation of Brauer groups of Zariski surfaces? In the paper [2] he computes the Brauer group as trivial, but this contradicts the main result of [1].

(2) Let S be a nonsingular projective surface such that $\operatorname{Pic} S \cong \mathbb{Z}$. We ask whether $\operatorname{Pic} S$ is generated by the divisor class of an irreducible curve $C \subseteq S$. The question arises because this was part of the hypotheses of Theorem 1.1 of [4]. Certainly we can assume S is minimal.

   (a) Some sufficient conditions are: Let $L = \mathcal{O}(D)$ be a generator for $\operatorname{Pic} S$. If $|D|$ is not composite with a pencil, then the corresponding morphism to projective space $S \to \mathbb{P}^N$ has image whose dimension is 2. By Bertini's theorem, there is an irreducible curve C in $|D|$.

   (b) If there is an irreducible curve C such that $S - C$ is factorial, then $\operatorname{Pic} S$ is generated by C.

   (c) If $\kappa(S) = -\infty$, then $S \cong \mathbb{P}^2$ and it is obvious.

   (d) If $\kappa(S) = 0$, the only possibility is that S is a K3 surface. Is $\operatorname{Pic} S$ generated by an irreducible curve, for a K3 surface?

   (e) How about for an elliptic surface S, $\kappa(S) = 1$, $\operatorname{Pic} S = \mathbb{Z}$?

   (f) How about for a surface S of general type, $\kappa(S) = 2$, $\operatorname{Pic} S = \mathbb{Z}$?

   (g) For hyperelliptic surfaces, $\beta_1 = 2$ so $\operatorname{Pic} S \neq \mathbb{Z}$. For abelian surfaces $\beta_1 = 4$, $\operatorname{Pic} S \neq \mathbb{Z}$. For Enriques surfaces, $\operatorname{Pic} S \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2$.