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On the Stochastic Adaptive Control of an Investment Model with Transaction Fees

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Abstract

In this paper we consider a stock market model where the investor can choose between a risky asset (stock) and a risk-free asset (bond) and there is a brokerage fee for selling the stocks. For the known model Taksar, Klass and Assaf describe the model in terms of the ratio of the wealth in stocks and in bonds and obtain an optimal control when it is desired to maximize the expected rate of growth of the wealth. For this model we assume that the parameters that govern the behavior of the value of the stock are unknown. A family of strongly consistent estimators is given and used to define a certainty equivalence adaptive control to solve the stochastic adaptive control problem. The solution is justified by numerical simulation for some specific parameters and shows that the family of numerically computed estimators approaches the true value and the family of costs using the numerically computed certainty equivalence adaptive control tends to the optimal cost.

1 Introduction and Model Description

Let us consider a market model, where an investor has two investment opportunities: one is risk free (bond) and the other one is risky (stock). Since Merton [10] first applied stochastic calculus to solve a consumption and investment problem, many extensions and variations of the problem have appeared in the literature. We focus our attention on a problem discussed by Taksar et al. in [12].

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space and let $\mathbf{F} = \{\mathcal{F}(t), t \geq 0\}$ be an augmented non-decreasing, right-continuous family of sub-σ-fields of $\mathcal{F}$. Furthermore let $\{W(t), t \geq 0\}$ be a real-valued standard Brownian motion adapted to $\mathbf{F}$. The evolution of the bond is given by the differential equation

$$dB(t) = rB(t)dt \quad B(0) = B_0 > 0 \tag{1.1}$$

and we model the evolution of the stock by the stochastic differential equation

$$dS(t) = S(t)(\mu + \sigma^2/2)dt + \sigma S(t)dW(t) \quad S(0) = S_0 \tag{1.2}$$

where $r > 0$ is the interest rate of the bond, $\mu + \sigma^2/2$ is the expected rate of return of the stock and σ is the volatility of the stock.

The control problem consists of the maximization of the expected rate of growth

$$E\{\liminf_{t \to \infty} [\ln Y(t)/t]\} \tag{1.3}$$

of the total wealth $Y(t)$ defined by

$$Y(t) = S(t) + B(t), \tag{1.4}$$

by means of trading between the two assets.

This problem has been solved by Karatzas [7] for the case in which there are no fees for transactions. Taksar et al. give a solution of the problem in the case in which a fee has to be paid by the investor whenever stocks are sold. The way in which this transaction fee is modeled is the following: Whenever stocks with the value of x are sold, the bond increases by λx, where $\lambda \in (0, 1)$.

The controlled processes $S(t)$ and $B(t)$ are then described by

$$dB(t) = rB(t)dt + \lambda dU(t) - dZ(t) \tag{1.5}$$

$$dS(t) = S(t)(\mu + \sigma^2/2)dt + \sigma S(t)dW(t) + dZ(t) - dU(t) \tag{1.6}$$

where $U(t)$ denotes the cumulative amount of money transferred from S to B up to time t, and $Z(t)$ denotes the cumulative amount transferred from B to S up to time t $(U(0) = Z(0) = 0)$.

In [12] the authors give an optimal policy for this problem that consists in keeping the quotient of the wealth invested in the stock and the wealth invested in the bond inside of an optimal interval by applying control whenever the value of the quotient reaches one of the endpoints of the interval.

A consumption and investment model with transaction fees where the goal is to maximize the discounted utility from consumption is solved in [2]; the solution that is obtained is similar to the solution in [12] just described. In fact the optimal policy keeps the processes $S(t)$ and $B(t)$ inside a wedge in the state space by reflecting at the boundaries.

In this paper we describe in sections 2 and 3 the model and the existence of the optimal control policy that are given in [12]. Then we explicitly construct the optimal control policy and test it through simulation. In section 4 we assume that the investor does not know the values of the parameters μ and σ in (1.2). Since the optimal interval (and therefore the optimal policy) depends explicitly on these values, the investor has to estimate these parameters. A strongly consistent family of estimators for μ is given, and it is shown that the value of σ is known to the controller after any positive interval of time: a formula for its computation is given. Then an adaptive control policy is defined using these estimates. This policy updates the endpoints of the interval, in which the quotient is to be held, at discrete moments of time according to the present value of the estimates. It is then shown that this adaptive control policy is both self-tuning and self-optimizing. Finally numerical results are given first to illustrate the behavior of the adaptive control policy and then to analyse the case in which the investor can observe prices and trade only at discrete moments of time.

2 Reformulation of the Problem

Following [12] we reformulate the control problem in terms of the ratio of the stock and the bond. Let

$$X(t) = \frac{S(t)}{B(t)}, \tag{2.1}$$

and define new control functionals

$$L(t) = \int_0^t S(s)^{-1} dU(s) \quad \text{and} \quad R(t) = \int_0^t B(s)^{-1} dZ(s), \tag{2.2}$$

which represent the cumulative percentage of stocks and bonds withdrawn up to time t.

Setting $X_0 = S_0/B_0$ the process $\{X(t), t \geq 0\}$ can be described by the stochastic differential equation

$$dX(t) = \sigma X(t) dW(t) + aX(t)dt - j(X(t))dL(t) + k(X(t))dR(t) \tag{2.3}$$

$$X(0) = X_0$$

where

$$j(x) = x + \lambda x^2, \quad k(x) = 1 + x, \tag{2.4}$$

and

$$a = (\mu + \sigma^2/2) - r \tag{2.5}$$

Taksar et al. show that the maximization of (1.3) is equivalent to the minimization of

$$J(L,R) = \limsup_{t\to\infty} t^{-1} E\{\int_0^t h(X(s))ds + \int_0^t g(X(s))dL(s)\} \tag{2.6}$$

where

$$h(x) = \frac{\sigma^2}{2} x^2/(x+1)^2 - ax/(x+1), \tag{2.7}$$
$$g(x) = (1-\lambda)x(x+1) \tag{2.8}$$

Remark 2.1 Note that

$$E\{\liminf_{t\to\infty}[\ln Y(t)/t]\} = r - J(L,R). \tag{2.9}$$

Hence $-J(L,R)$ represents the gain in the rate of growth achieved by using the policy (L,R) in comparison to the interest rate r of the bond.

Definition 2.1 An *admissible control policy* for the reformulated problem is a pair of right-continuous, nondecreasing processes $(L(t), R(t))_{t\geq 0}$ such that

1. L and R are adapted to $\mathbf{F}$.
2. $$E\left[\int_0^t S(s)dL(s)\right] + E\left[\int_0^t B(s)dR(s)\right] < \infty \quad \forall t \geq 0 \tag{2.10}$$
3. (2.3) has a unique, nonnegative solution.

Remark 2.2 The function h(x) in (2.7) can be interpreted as the relative holding costs at the point x. Nevertheless h(x) can have negative values. The function g(x) in (2.8) represents the transaction costs that appear when the investor applies control at the point x.

Remark 2.3 In [12] it is shown that the problem is trivial if $a \leq 0$ or $a \geq \sigma^2$. In the first case the optimal policy is to invest everything in the bond ($X = 0$) and to apply no further control. In the second case the costs are minimized by investing everything in the stock ($X \to \infty$). For the nontrivial case where $0 < a < \sigma^2$, the holding costs are minimized at the point x^* for which the derivative of the holding cost function $h'(x) = 0$. This happens iff $x^*/(x^*+1) = a/\sigma^2$, which is equivalent to

$$x^* = \frac{\mu - r + \sigma^2/2}{r - \mu + \sigma^2/2} \tag{2.11}$$

3 Optimal Control Policy

31 Existence of the Optimal Control Policy

For the control problem (2.3) and (2.6) an optimal control policy is described and its existence is proven in [12]. In this subsection we want to summarize the main results of that paper.

Theorem 3.1 (Taksar, Klass, Assaf)

1. *For the control problem formulated in (2.3) and (2.6) there is an unique optimal policy* (L^*, R^*).

2. *The optimal policy* (L^*, R^*) *is a double bound control policy that keeps the process* $X(t)$ *inside an interval* $[A, B]$, $0 < A \leq B$, *where* A *and* B *can be determined by the equations*

$$B^{-\alpha} \int_A^B 2\sigma^{-2}[h(A) - h(y)] y^{\alpha-2} dy = G(B) \tag{3.1}$$

and

$$B = \lambda^{-1}((\alpha - 1)A + \alpha)/((2-\alpha)A + 1 - \alpha) \tag{3.2}$$

where $\alpha = \dfrac{2\,a}{\sigma^2}$ *and* $G(x) = \frac{g(x)}{j(x)}$.

3. *For this optimal policy* (L^*, R^*) *the limit inferior in (1.3) can be replaced by limit and*

$$E\{\liminf_{t\to\infty} \frac{1}{t} \ln Y(t)\} \;=\; \lim_{t\to\infty} \frac{1}{t} \ln Y(t) \quad a.\ s. \tag{3.3}$$

4. *For the optimal control policy the limit superior in (2.6) is a limit and*

$$J(L^*, R^*) = h(A) \tag{3.4}$$

Remark 3.1 The optimal interval contains the optimal ratio x^* given in Remark 2.3. The optimal policy keeps $X(t)$ near x^* in such a way that the holding costs are "small" without applying too much control.

32 Construction of the Optimal Control Policy

The construction of the unique optimal control policy follows [6]. The authors consider a diffusion process $\xi(t)$ to be kept in the interval $[0,1]$. They show that there are two nondecreasing, **F**-measurable control functionals $\zeta_1(t)$ and $\zeta_2(t)$ that reflect $\xi(t)$ at the endpoints of the interval such that, if $\xi(0) \in [0,1]$ and $a(t,s)$, $\sigma(t,s)$ are continuous in t and s, the functionals $\xi(t)$, $\zeta_1(t)$ and $\zeta_2(t)$ are related by

$$\xi(t) = \xi(0) + \int_0^t a(s, X(s))ds + \int_0^t \sigma(s, X(s))dw(s) + \zeta_1(t) - \zeta_2(t) \quad (3.5)$$

where

$$\zeta_1(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \chi_{(-\infty,\epsilon]}(\xi(s))\sigma^2(s,\xi(s))ds, \quad (3.6)$$

$$\zeta_2(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \chi_{[1-\epsilon,\infty)}(\xi(s))\sigma^2(s,\xi(s))ds. \quad (3.7)$$

and

$$\chi_S(x) = \begin{cases} 1 & : \quad x \in S \\ 0 & : \quad x \notin S \end{cases} \quad (3.8)$$

The limits in (3.6) and (3.7) exist in the sense of convergence in probability. Using an affine transformation of the processes $\xi(t)$, $\zeta_1(t)$ and $\zeta_2(t)$ into the interval [A,B], calling these new processes $X(t)$, $r(t)$ and $l(t)$ resp., and setting $a(t,s) = as$, $\sigma(t,s) = \sigma s$ we obtain

$$X(t) = X(0) + \int_0^t aX(s)ds + \int_0^t \sigma X(s)dw(s) + r(t) - l(t) \quad (3.9)$$

where

$$r(t) = \lim_{\epsilon \to 0} \int_0^t \chi_{(-\infty,A+\epsilon]}(X(s)) \frac{\sigma^2(B-A)}{2\epsilon} X(s)ds, \quad (3.10)$$

$$l(t) = \lim_{\epsilon \to 0} \int_0^t \chi_{[B-\epsilon,\infty)}(X(s)) \frac{\sigma^2(B-A)}{2\epsilon} X(s)ds. \quad (3.11)$$

Similar to (3.6) and (3.7) the limits in (3.10) and (3.11) exist in the sense of convergence in probability. The control is a double bound policy, which reflects the process at the endpoints A and B of the interval by giving an infinitely large push during an infinitely small time interval. The control is applied when the process $X(t)$ hits one of the bounds. We can rewrite (3.9) as

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t aX(s)ds + \int_0^t \sigma X(s)dW(s) \\
&+ \lim_{\epsilon\to 0}\int_0^t k(X(s))dR_\epsilon(s) - \lim_{\epsilon\to 0}\int_0^t j(X(s))dL_\epsilon(s) \qquad (3.12)
\end{aligned}
$$

where

$$
dR_\epsilon(s) = \frac{X(s)}{k(X(s))}\chi_{(-\infty,A+\epsilon]}(X(s))d\nu_\epsilon, \qquad (3.13)
$$

$$
dL_\epsilon(s) = \frac{X(s)}{j(X(s))}\chi_{[B-\epsilon,\infty)}(X(s))d\nu_\epsilon \qquad (3.14)
$$

and

$$
d\nu_\epsilon = \frac{\sigma^2(B-A)}{2\epsilon}ds \qquad (3.15)
$$

and $k(x)$ and $j(x)$ are as defined in (2.4).

Finally we define the optimal control limit policy (L^*, R^*) by

$$
R^* = \lim_{\epsilon\to 0} R_\epsilon, \qquad L^* = \lim_{\epsilon\to 0} L_\epsilon \qquad (3.16)
$$

where the limits in (3.16) exist in the sense of weak* convergence. Since $X(t)$ is bounded and the functions $k(x)$ and $j(x)$ are continuous, we can interchange limit and integration in (3.12) by the definition of weak* convergence and obtain

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t aX(s)ds + \int_0^t \sigma X(s)dW(s) \\
&+ \int_0^t k(X(s))dR^*(s) - \int_0^t j(X(s))dL^*(s) \qquad (3.17)
\end{aligned}
$$

which is the integral version of (2.3).

33 Numerical Simulation of the Optimal Control x Problem

To simulate the process $X(t)$ we discretize the trajectory of $X(t)$ in a finite interval $[0, T]$ at $(n+1)$ equidistant points X_k with stepsize $\Delta t = T/n$ using Milstein's method (see [9]) such that $X_0 = X(0)$ and X_k is a numerical approximation of $X(k\Delta t)$, $k \le n$.

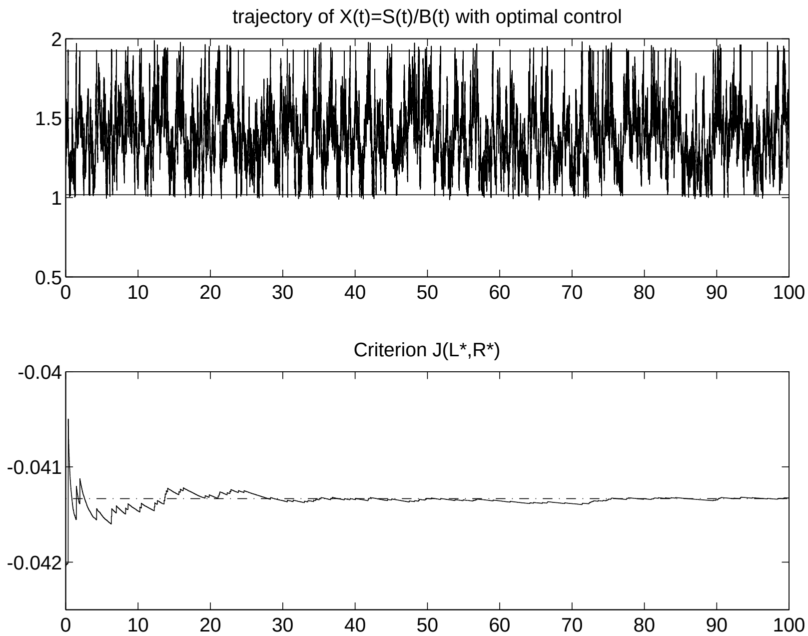

Fig. 1: Numerical Simulation of the Optimal Control Problem

The continuous time control policy (L*,R*) described in (3.13) - (3.16) is replaced by the following: Fix a sufficiently small $\epsilon > 0$ and discretize ν and the controls R_ϵ, L_ϵ by replacing dt by Δt.

There is apparently no explicit solution for the differential equation (2.3). Therefore the only possibility to check the accuracy of the discretization and simultaneously to verify the theoretical results of [12] is to examine the criterion J in (2.6). To do this we define for the optimal policy (R^*, L^*)

$$C(t) = t^{-1} \int_0^t h(X(s))ds + \int_0^t g(X(s))dL^*(s) \tag{3.18}$$

From Theorem 3.1 we can conclude that

$$\lim_{t \to \infty} C(t) = J(L^*, R^*) = h(A) \quad \text{a.s.} \tag{3.19}$$

By using the numerical solution $X_k, k = 0, 1, \ldots$ instead of X itself and approximating the integrals appearing in () by rectangular quadrature we get a numerical approximation of $C(t)$. A computational example of the trajectories of $X(t)$ and $C(t)$ is shown in Figure 1. Here the parameters are $\mu = .1$, $r = .08$, $\sigma = .5$, $\lambda = .99$, $\Delta t = .001$ and $\epsilon = .0005$. The

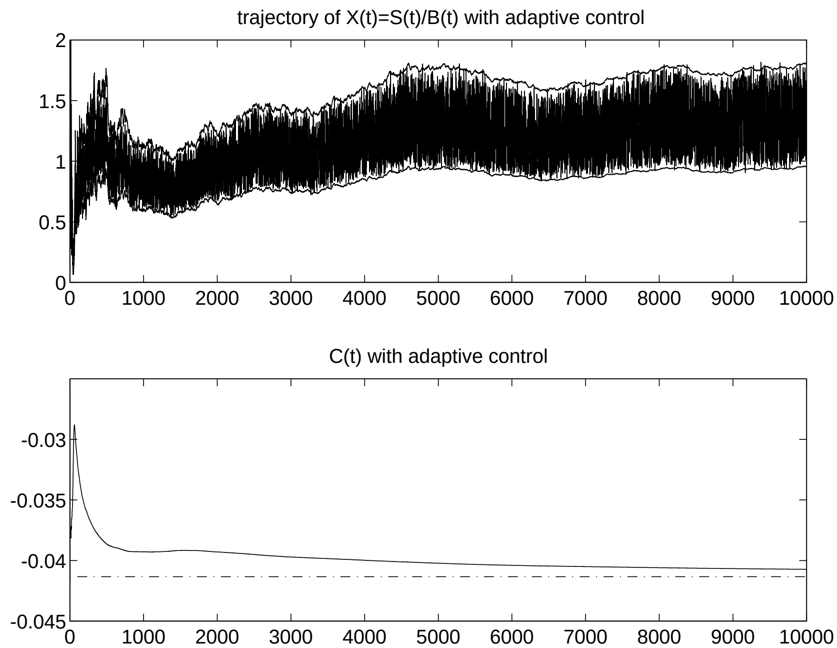

Fig. 2: Numerical Simulation of the Adaptive Control Problem

process $\{X(t), t \geq 0\}$ is kept inside the interval $[A, B]$ and the function $C(t)$ converges to $h(A)$ as stated above.

4 System Identification and Adaptive Control

41 Parameter Estimation

For the estimation of the unknown parameter a in the case of continuous observation we use an estimator that is similar to the estimators used in [1], [3], [11], and [13].

Definition 4.1 For every $t > 0$ and every admissible control (L, R), let

$$\widehat{a}_t = \begin{cases} a_1 & : \quad t < t_0 \\ \dfrac{1}{t}\displaystyle\int_0^t \frac{dX(s) + j(X(s))dL(s) - k(X(s))dR(s)}{X(s)} & : \quad t \geq t_0 \end{cases} \tag{4.1}$$

where $t_0 > 0$ is fixed and a_1 is an arbitrary initial estimate.

Theorem 4.1 *The family of estimators* $\{\widehat{a}_t, t > 0\}$ *defined in (4.1) is strongly consistent, i.e.*

$$\lim_{t\to\infty} \widehat{a}_t = a \quad a.\ s. \tag{4.2}$$

Proof. Using (2.3) we obtain

$$\begin{aligned} \widehat{a}_t &= \frac{1}{t}\int_0^t \frac{dX(s) + j(X(s))dL(s) - k(X(s))dR(s)}{X(s)} \\ &= \frac{1}{t}\int_0^t \frac{aX(s)ds + \sigma X(s)dW(s)}{X(s)} \\ &= a + \sigma\frac{W(t)}{t} \end{aligned} \tag{4.3}$$

and the result (4.2) follows from the strong law of large numbers for Brownian motion.

□

Using the relationship $\widehat{a}_t = a + \sigma W(t)/t$ in (4.3) we can give some more properties of $\widehat{a}_t$ (see also [1] and [4]) :

Proposition 4.2 *For any admissible control* (L, R)*, the family of estimators* $\{\widehat{a}_t, t > 0\}$ *has the following properties*

1. $\widehat{a}_t$ *is normally distributed, i.e.*

$$\widehat{a}_t \sim \mathcal{N}(a, \frac{\sigma^2}{t}), \quad t \geq t_0 \tag{4.4}$$

2. *The rate of convergence is*

$$\forall \epsilon > 0 \qquad P\left\{\limsup_{t\to\infty} \frac{|\widehat{a}_t - a|}{\sigma\sqrt{2\,t^{-1}\ln\ln t}} \leq 1 + \epsilon\right\} = 1 \tag{4.5}$$

Proof. The first property follows directly from $W(t) \sim \mathcal{N}(0, t)$. The second property is a consequence of the law of the iterated logarithm for Brownian motion (see [8]):

$$P\left\{\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\ln\ln t}} = 1\right\} = 1 \tag{4.6}$$

□

We now want to describe, how the investor can determine the volatility σ of the process $X(t)$. Since the adaptive control policy given in section 42 depends on σ^2 (not on σ), we give a formula for its computation:

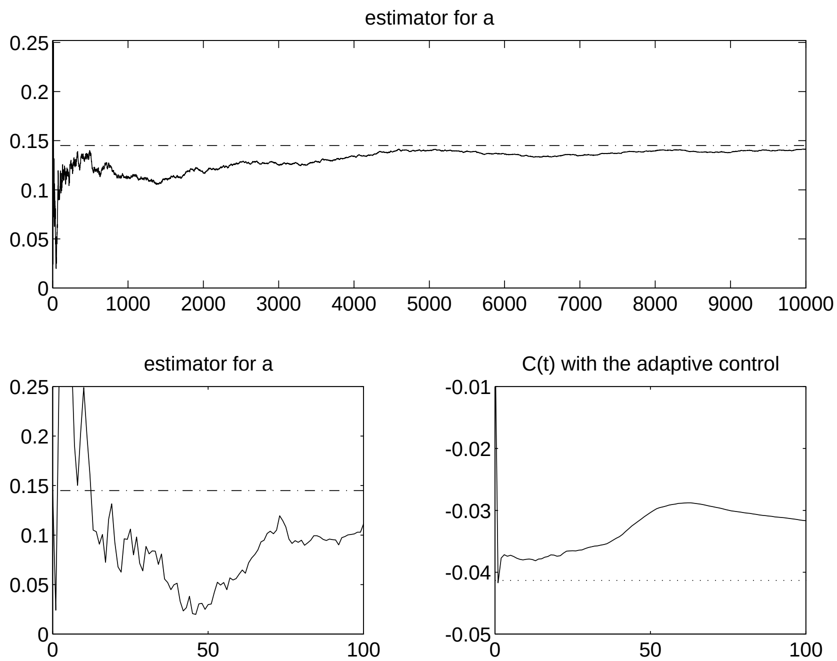

Fig. 3: Numerical Simulation of the Adaptive Control Problem

For each interval $[0, t]$, $t > 0$ we define a sequence of partitions $\{p_t^n, n \geq 1\}$ where for each n, $p_t^n = \{t_0, t_1, \ldots, t_k | 0 = t_0 \leq t_1 \leq \ldots \leq t_k = t\}$ with mesh $|p_t^n| = \max_{i=1}^k |t_i - t_{i-1}|$, such that $\sum_{n=1}^\infty |p_t^n| < \infty$. For each p_t^n set

$$\widehat{\sigma}^2(p_t^n) = \frac{1}{t} \sum_{i=1}^k \frac{\left[X(t_i) - X(t_{i-1}) + \int_{t_i-1}^{t_i} j(X(s))dL(s) - \int_{t_i-1}^{t_i} k(X(s))dR(s)\right]^2}{\left[X(t_{i-1})\right]^2} \tag{4.7}$$

According to the properties of the quadratic variation of a Brownian motion (see [8]) it follows that $\forall t > 0$

$$\lim_{n\to\infty} \widehat{\sigma}^2(p_t^n) = \sigma^2 \quad \text{a.s.} \tag{4.8}$$

i.e. in the case of continuous observation the exact value of σ^2 can be determined by the investor after any arbitrarily short time $t > 0$.

42 Adaptive Control

In the adaptive control of the problem (2.3) and (2.6) we assume that the parameter a is unknown. Since we can get the exact value of σ^2 after an arbitrary small amount of time, we can assume that σ^2 is known. Using

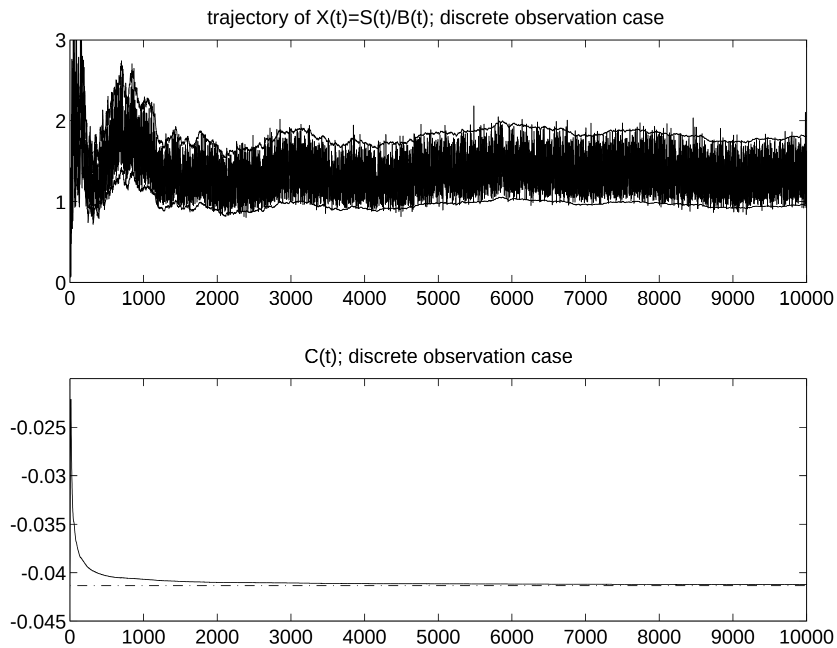

Fig. 4: Discrete Observation Case

the estimator for a given in section 41, we define an adaptive double bound control policy for the investor based on the past observations.

Definition 4.2 Let t_0 from Definition 4.1 be equal to 1 and define for every $t \geq 0$ the adaptive bounds

$$\hat{A}_t = \begin{cases} \sum_{k=1}^{\infty} \chi_{(k,k+1]}(t)\, A(\hat{a}_k) & t > 1 \\ A_0 & 0 \leq t \leq 1 \end{cases} \tag{4.9}$$

$$\hat{B}_t = \begin{cases} \sum_{k=1}^{\infty} \chi_{(k,k+1]}(t)\, B(\hat{a}_k) & t > 1 \\ B_0 & 0 \leq t \leq 1 \end{cases} \tag{4.10}$$

where $A(\hat{a}_k)$ and $B(\hat{a}_k)$ are the solutions of the system of equations (3.1)-(3.2) where $\alpha = \dfrac{2\,\hat{a}_k}{\sigma^2}$, and A_0, B_0 are arbitrary constants such that $0 < A_0 < B_0$, and $\widehat{a}_k$ is given by (4.1).

We obtain an adaptive control policy by replacing A and B in (3.13) - (3.15) by $\hat{A}_t$ and $\hat{B}_t$ respectively.

Theorem 4.3 *The adaptive control policy that uses the adaptive bounds defined in (4.9), (4.10) is self-tuning and self-optimizing.*

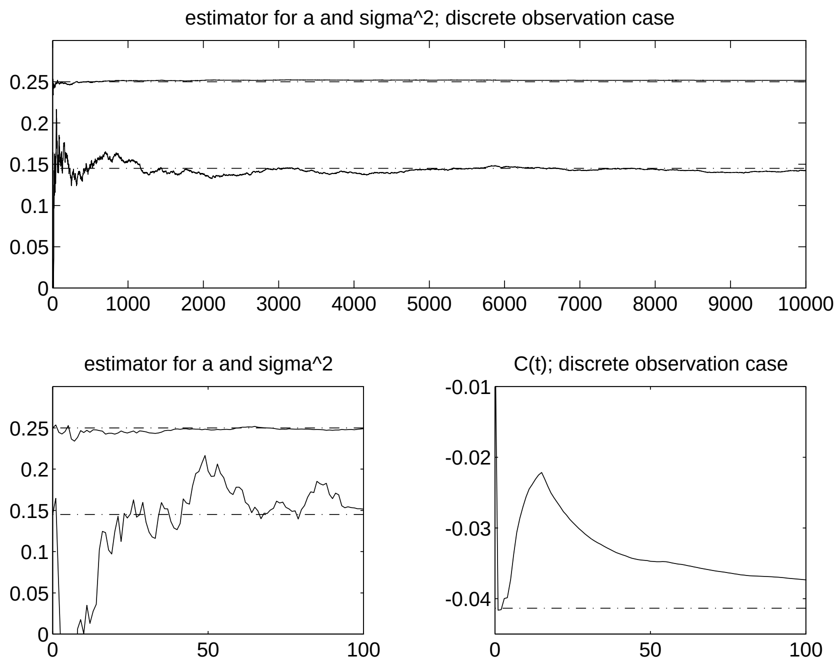

Fig. 5: Discrete Observation Case

Proof. The strong consistency of $\widehat{a}_t$ and the continuity of A and B with respect to α in (3.1) and (3.2) implies that the control is self-tuning. The boundedness of the trajectory of $X(t)$ easily implies the self-optimality.

□

43 Numerical Simulation of the Adaptive Control x Problem

We continue the presentation of the numerical results initiated in subsection 33. In this subsection we assume that the parameter μ in (1.2) is unknown and simulate the process controlled by the adaptive policy defined in subsection 42. The values of the parameters are the same as in subsection 33.

Figure 2 shows on the top graph the trajectory of $X(t)$ with the adaptive boundaries $\hat{A}_t$ and $\hat{B}_t$, and on the bottom graph the trajectory of $C(t)$ using the adaptive control as a solid line and the theoretical limit $h(A)$ as a dashed line. It appears, that in the long run $C(t)$ converges as expected.

In Figure 3 the numerical behavior of the estimator defined in definition 4.1 is given. This graph is consistent with the theoretical results for strong consistency and the rate of convergence. The other two graphs show the short time behavior of the estimator and $C(t)$.

44 Numerical Simulation of the Adaptive Control x Problem with Discrete Observation

The final simulation considers the case in which the investor can observe $X(t)$ and apply control only at discrete times. To implement such a case we let the stepsize for the simulation be $\Delta t = 10^{-4}$ and observe the trajectory only at times $t_k = k10^{-2}$ $k = 0, 1, 2, \ldots$, i.e. we observe one point out of 100. If μ and σ are unknown, then we need to estimate the values of both parameters using discretized versions of (4.1) and (4.7). The results are shown in figures 4 and 5. In figure 4 we have the graph of the trajectory $X(t)$ and the adaptive bounds and the graph of $C(t)$ and in figure 5 the graph of the estimators for a and σ^2.

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