Ulam Quaterly – Volume 2, Number 4, 1994

A Generalization of the Littlewood-Paley Inequality and Some Other Results Related to Stochastic Partial Differential Equations

N.V. Krylov[1]

University of Minnesota
School of Mathematics
Minneapolis, Minnesota

Abstract

We prove a "parabolic" type Littlewood-Paley inequality and a theorem relating norms of functions in H^n_p to norms of their products with elements of a partition of unity.

Introduction

In the author's work [K] on L_p-theory of stochastic partial differential equations some results concerning spaces of Bessel potentials turned out to be necessary. The author could not find them in the literature in the necessary form, and at the same time they look useful for other applications. These are the reasons to publish them in a separate paper.

Recall (see, for instance, [T1], [T2]) that the space $H^n_p = H^n_p(R^d)$, $p \in (1,\infty)$, $n \in (-\infty,\infty)$, of Bessel potentials is defined as the closure of $C^\infty_0 = C^\infty_0(R^d)$ with respect to the norm

$$||u||_{n,p} = ||(I-\Delta)^{n/2}u||_p, \text{ where } ||h||_p = (\int_{R^d} |h(x)|^p \, dx)^{1/p}.$$

[1]Supported in part by NSF Grant DMS–9302516.

We are presenting here two results. The first one is related to the following problem. Consider the simplest one-dimensional SPDE

$$du(t,x) = \frac{1}{2}u_{xx}(t,x)\,dt + g(t,x)\,dw_t \quad t>0, u(0,x)=0,$$

where w_t is a one-dimensional Wiener process. The solution of this problem is known to be

$$u(t,x) = \int_0^t T_{t-s}g(s,\cdot)(x)\,dw_t,$$

where $T_t h(x) = Eh(x+w_t)$. If g is non random,

$$E\int_0^T ||u(t,\cdot)||_{1,p}^p\,dt = N(p)\int_0^T\int_{R^d}[\int_0^t|(I-\Delta)^{1/2}T_{t-s}g(s,\cdot)(x)|^2\,ds]^{p/2}\,dx\,dt,$$

and in order to prove that $u \in H_p^1$ we have to estimate the last integral. Such estimates are considered in Section 1.

In Section 2 we prove the so-called uniform localization theorem for H_p^n and some facts related to it. This theorem is needed when one wants to prove existence theorems for equations with variable coefficients on the basis of existence theorems for equations with constant coefficients. Roughly speaking the theorem says that the H_p^n-norm of a function can be expressed in terms of norms of products of this functions with elements of a smooth partition of unity. Such a theorem for $n \geq 0$ is proved in [S], and the general case is essentially covered by Theorem 2.4.7 [T2]. In [K] we need the corresponding result, for example, for n slightly less than $-3/2$. Although our version of the uniform localization theorem and the important Lemma 2.1 can be obtained through a series of references, sometimes accompanied by words "as in the proof...", to results on function spaces more general than H_p^n from [T2], we think that our proofs have their merits too, being based only on very well known (to specialists in PDEs) properties of H_p^n.

Acknowledgements

The author is sincerely grateful to E. Fabes, C. Kenig, W. Littman and H. Triebel for very helpful advice and discussions and S. Lapic for pointing out several mistakes and misprints in the last draft of the article.

1. A generalization of the Littlewood-Paley inequality

Let $T_t, t \geq 0$, be the semigroup corresponding to the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$$

in R^d. The classical Littlewood-Paley inequality says that for any $p \in (1,\infty)$ and $g \in L_p$ it holds that

$$\int_{R^d}[\int_0^\infty |\nabla T_t g(x)|^2\,dt]^{p/2}\,dx \leq N||g||_p^p, \tag{1.1}$$

where the constant N depends only on d, p.

Here we want to generalize this fact by proving

Theorem 1.1 *Let H be a Hilbert space, $p \in [2,\infty)$, $-\infty \le a < b \le \infty$, $g \in L_p((a,b)\times R^d, H)$. Then*

$$\int_{R^d}\int_a^b [\int_a^t |\nabla T_{t-s} g(s,\cdot)(x)|_H^2\,ds]^{p/2}\,dt\,dx \le N\int_{R^d}\int_a^b |g(t,x)|_H^p\,dt\,dx, \tag{1.2}$$

where the constant N depends only on d, p.

Remark 1.1 The Littlewood-Paley inequality (1.1) follows directly from (1.2) if $p \ge 2$. Indeed, take $a = 0, b = 2, g(s,x) = g(x)$. Then the left-hand side of (1.2) equals

$$\int_{R^d}\int_0^2 [\int_0^t |\nabla T_s g(t-s,\cdot)(x)|_H^2\,ds]^{p/2}\,dt\,dx$$

$$\ge \int_{R^d}\int_1^2 [\int_0^t |\nabla T_s g(x)|_H^2\,ds]^{p/2}\,dt\,dx \ge \int_{R^d}[\int_0^1 |\nabla T_s g(x)|_H^2\,ds]^{p/2}\,dx.$$

Thus, from (1.2) it follows that

$$\int_{R^d}[\int_0^1 |\nabla T_s g(x)|_H^2\,ds]^{p/2}\,dx \le N\int_{R^d}|g(x)|_H^p\,dx,$$

and a standard argument based on the self-similarity allows us to replace the upper limit 1 by infinity keeping the *same* constant N.

This gives (1.1) for $p \ge 2$ and for Hilbert space valued g. It is a standard fact that from the Hilbert-space version of (1.1) for $p \ge 2$ the same inequality follows for $p \in (1,2)$ by duality.

It is also well known that having (1.1) proved for $p \in (1,2]$, the case $p \in [2,\infty)$ can be treated by duality argument. In contrast with this "symmetry" we have

Remark 1.2 For $p \in (1,2)$ estimate (1.2) is not true even if $d = 1$, $H = R$. Indeed, take $a = 0$, a finite b, $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, $g(s,x) = \phi(x)e^{-\lambda s}$ where $\lambda > 0$. Then

$$\nabla T_t g(s,x) = \sqrt{t}\,\frac{\partial}{\partial x}[t^{-1/2}\phi(x/\sqrt{t}) * \phi(x)]e^{-\lambda s}$$

$$= \sqrt{t}\,\frac{\partial}{\partial x}[(t+1)^{-1/2}\phi(x/\sqrt{t+1})]e^{-\lambda s} = \sqrt{t}\,e^{-\lambda s}(t+1)^{-1}\phi'(x/\sqrt{t+1}),$$

and as easy to see the limit as λ goes to infinity of the product of the left-hand side of (1.2) and $\lambda^{p/2}$ equals

$$\lim_{\lambda\to\infty}\int_R\int_0^b [\lambda\int_0^t \frac{x^2}{2\pi(t-s+1)^3}e^{-2\lambda s}e^{-\frac{1}{t-s+1}x^2}\,ds]^{p/2}\,dt\,dx. \tag{1.3}$$

Furthermore for $\lambda \geq 1$

$$\int_0^t \frac{\lambda x^2}{2\pi(t-s+1)^3} e^{-2\lambda s} e^{-\frac{1}{t-s+1}x^2}\, ds = \int_0^{t/2} + \int_{t/2}^t$$

$$\leq \frac{Nx^2}{(t+2)^3} e^{-\frac{1}{t+1}x^2} \int_0^{t/2} \lambda e^{-2\lambda s}\, ds + e^{-\lambda t} N x^2 e^{-\frac{2}{t+2}x^2} \leq \frac{Nx^2}{(t+2)^3} e^{-\frac{2}{t+2}x^2}.$$

This allows us to evaluate the limit in (1.3) by using the dominated convergence theorem. We see that the limit equals

$$\int_R \int_0^b \big[\frac{x^2}{4\pi(t+1)^3} e^{-\frac{1}{t+1}x^2}\big]^{p/2}\, dt\, dx,$$

which is finite and non zero.

Thus the left-hand side of (1.2) is of order $\lambda^{-p/2}$. At the same time the right-hand side of (1.2) is of order λ^{-1}, and $\lambda^{-p/2}$ is much bigger than λ^{-1} if $p \in (1,2)$ and $\lambda \to \infty$.

Proof of Theorem 1.1. First note that by considering $g(t,x)I_{(a,b)}(t)$ instead of $g(t,x)$ we can reduce the general case to the one in which $a = -\infty, b = \infty$. Therefore, only this case we consider below. Next, for $p = 2$ the application of the Fourier transformation shows that the left-hand side in (1.2) equals

$$\int_{R^d} \int_{-\infty}^{\infty} \int_0^t |\xi|^2 e^{-|\xi|^2(t-s)} |\tilde{g}(s,\xi)|_H^2\, ds\, dt\, d\xi = \int_{R^d} \int_{-\infty}^{\infty} |\tilde{g}(s,\xi)|_H^2\, ds\, d\xi,$$

what in its turn equals the right-hand side of (1.2). This proves (1.2) for $p = 2$ and shows that if we introduce the operator P by the formula

$$Pg(t,x) = [\int_{-\infty}^t |\nabla T_{t-s} g(s,\cdot)(x)|_H^2\, ds]^{1/2},$$

then P is a bounded (actually, isometric) operator from $L_2(R^{d+1}, H)$ into $L_2(R^{d+1}, R)$.

Due to the parabolic sharp inequality ($||f||_p \leq N||f^{\#}||_p$, see [B]) and the Marcinkiewicz interpolation theorem the parabolic version of the Stampacchia interpolation theorem is available, and we will prove our theorem if we prove that P is a bounded operator from $L_\infty(R^{d+1}, H)$ into $BMO(R^{d+1}, R)$. More precisely, it suffices to show that if $|g(t,x)|_H \leq 1$ for all $(t,x) \in R^{d+1}$ and g vanishes for $|t| + |x|$ large enough, then for any set $Q = (t_0, t_0 + r^2) \times \{x : |x - x_0| \leq r\}$ with $t_0 \in R$, $r > 0$, $x_0 \in R^d$ there exist a constant $g_Q \in R$ (depending on g, Q) and an absolute constant N such that

$$\int_Q |Pg(t,x) - g_Q|^2\, dt\, dx \leq N|Q|. \tag{1.4}$$

A shift of the origin shows that we can always take $t_0 = 0, x_0 = 0$ in (1.4). Furthermore, if we replace $g(t,x)$ by $g(tr^{-2}, xr^{-1})$ then we see that without loss of generality we can take $r = 1$ in (1.4). Thus, let $t_0 = 0, x_0 = 0$, $r = 1$. Also observe that for $t \in (0,1)$

$$Pg(t,x) \le [\int_{-\infty}^{-1} |\nabla T_{t-s} g(s,\cdot)(x)|_H^2 \, ds]^{1/2} + [\int_{-1}^{t} |\nabla T_{t-s} g(s,\cdot)(x)|_H^2 \, ds]^{1/2}$$

$$:= P_1 g(t,x) + P_2 g(t,x).$$

On the other hand obviously $P_1 g \le Pg$. It follows that for any constant g_Q

$$|Pg(t,x) - g_Q| \le |P_1 g(t,x) - g_Q| + |P_2 g(t,x)|. \tag{1.5}$$

Further, for $s < t$

$$|\nabla T_{t-s} g(s,\cdot)|_H^2 = \frac{1}{2}\Delta(|T_{t-s} g(s,\cdot)|_H^2) - \frac{\partial}{\partial t}(|T_{t-s} g(s,\cdot)|_H^2).$$

It follows that

$$\int_{-1}^{t} |\nabla T_{t-s} g(s,\cdot)|_H^2 \, ds = \frac{1}{2}\Delta \int_{-1}^{t} |T_{t-s} g(s,\cdot)|_H^2 \, ds - \frac{\partial}{\partial t}\int_{-1}^{t} |T_{t-s} g(s,\cdot)|_H^2 \, ds$$

$$+|g(t,\cdot)|_H^2.$$

Using this and taking a function $\zeta \in C^\infty(R^d)$ such that $\zeta(x) = 0$ if $|x| \ge 2$, $\zeta(x) = 1$ if $|x| \le 1$, we see that

$$\int_Q |P_2 g(t,x)|^2 \, dt \, dx = \int_{|x|<1} \int_0^1 (\int_{-1}^{t} |\nabla T_{t-s} g(s,\cdot)(x)|_H^2 \, ds) \, dt \, dx$$

$$\le \int_{R^d} \int_0^1 \zeta (\int_{-1}^{t} |\nabla T_{t-s} g(s,\cdot)(x)|_H^2 \, ds) \, dt \, dx =$$

$$\int_{R^d} \int_0^1 \{\zeta |g(t,x)|_H^2 + (\frac{1}{2}\Delta\zeta) \int_{-1}^{t} |T_{t-s} g(s,\cdot)(x)|_H^2 \, ds\} \, dt \, dx$$

$$+ \int_{R^d} \zeta \{\int_{-1}^{0} |T_{-s} g(s,\cdot)(x)|_H^2 \, ds - \int_{-1}^{1} |T_{1-s} g(s,\cdot)(x)|_H^2 \, ds\} \, dx.$$

The last expression is obviously bounded by a constant independent of g (recall that $|g|_H \le 1$).

Owing to (1.5) it remains only to find an appropriate g_Q such that

$$\int_Q |P_1 g(t,x) - g_Q|^2 \, dt \, dx$$

is bounded by a constant independent of g. Actually, it turns out that as g_Q one can take any particular value of $P_1 g(t,x)$ in Q. Indeed, as we will see first derivatives of $P_1 g(t,x)$ are bounded on Q.

Let us first estimate the derivatives with respect to x. By the Minkowski inequality

$$|\frac{\partial}{\partial x^i}P_1 g(t,x)| \leq [\int_{-\infty}^{-1} |\nabla \frac{\partial}{\partial x^i} T_{t-s} g(s,\cdot)(x)|_H^2 \, ds]^{1/2}.$$

Here

$$\frac{\partial^2}{\partial x^i x^j} T_{t-s} g(s,\cdot)(x)$$

$$= \frac{1}{(2\pi(t-s))^{d/2}} \int_{R^d} [\frac{(x^i-y^i)(x^j-y^j)}{(t-s)^2} - \frac{\delta^{ij}}{t-s}] e^{-\frac{1}{2(t-s)}(x-y)^2} g(s,y)\, dy.$$

It follows that

$$|\nabla \frac{\partial}{\partial x^i} T_{t-s} g(s,\cdot)(x)|_H^2 \leq \frac{N}{(t-s)^2}\{\frac{1}{(t-s)^{d/2}} \int_{R^d} e^{-\frac{1}{t-s}(x-y)^2}\, dy\}^2 = \frac{N}{(t-s)^2},$$

and for $t \geq 0$

$$[\int_{-\infty}^{-1} |\nabla \frac{\partial}{\partial x^i} T_{t-s} g(s,\cdot)(x)|_H^2 \, ds]^{1/2} \leq N[\int_{-\infty}^{-1} \frac{1}{(t-s)^2}\, ds]^{1/2}$$

$$\leq N[\int_{-\infty}^{-1} \frac{1}{(-s)^2}\, ds]^{1/2} = N.$$

Thus, we get on Q the estimate of the gradient of $P_1 g(t,x)$ with respect to x. In like manner the first derivative of $P_1 g(t,x)$ with respect to t can be estimated, and the theorem is proved.

Remark 1.3 As explained in the introduction, in [K] we also need the following version of (1.2):

$$\int_{R^d} \int_a^b [\int_a^t |(I-\Delta)^{1/2} T_{t-s} g(s,\cdot)(x)|_H^2 \, ds]^{p/2} \, dt\, dx \leq$$

$$[1+(b-a)^{p/2}]N(d,p) \int_{R^d} \int_a^b |g(t,x)|_H^p \, dt\, dx.$$

To deduce this from (1.2) it suffices to observe that

$$(I-\Delta)^{1/2} T_{t-s} g = (I-\Delta) T_{t-s} (I-\Delta)^{-1/2} g = T_{t-s} R g - \frac{\partial}{\partial x^i} T_{t-s} P_i g,$$

where $R = (I-\Delta)^{-1/2}$, $P_i = (\partial/\partial x^i)R$ are bounded operators in L_p for any $p \in (1,\infty)$, and by the well-known estimate of norms of convolutions

$$\int_{R^d} \int_a^b [\int_a^t |T_{t-s} R g(s,\cdot)(x)|_H^2 \, ds]^{p/2} \, dt\, dx$$

$$\leq \int_{R^d}\int_a^b [\int_a^t T_{t-s}R|g(s,\cdot)|_H^2(x)\,ds]^{p/2}\,dt\,dx \leq$$

$$(\int_0^{b-a} T_t R1(0)\,dt)^{p/2}\int_{R^d}\int_a^b |g(t,x)|_H^p\,dt\,dx = (b-a)^{p/2}\int_{R^d}\int_a^b |g(t,x)|_H^p\,dt\,dx.$$

2. Uniform localization theorem for spaces of Bessel potentials

Our aim here is to prove the following result.

Theorem 2.1 *Let $p \in (1,\infty), n \in (-\infty,\infty)$, $\delta > 0$ and let $\zeta_k \in C^\infty$, $k = 1,2,3,\dots$. Assume that for any multi-index α and $x \in R^d$*

$$\sup_{x\in R^d}\sum_k |D^\alpha \zeta_k(x)| \leq M(\alpha), \quad \sum_k |\zeta_k(x)|^p \geq \delta, \tag{2.1}$$

where $M(\alpha)$ are some constants. Then there exists a constant $N = N(d,n,M,\delta)$ such that for any $f \in H_p^n$

$$||f||_{n,p}^p \leq N\sum_k ||\zeta_k f||_{n,p}^p, \quad \sum_k ||\zeta_k f||_{H_p^n}^p \leq N||f||_{H_p^n}^p. \tag{2.2}$$

To prove the theorem we need a lemma in which we denote $L_\lambda = \lambda I - \Delta$, and for the Green's function $G(x)$ of the operator $(I-\Delta)^m$, $m \geq 0$, $\lambda > 0$ and multi-indices β such that $|\beta| \leq 2m-1$ we define

$$G_{m\lambda\beta}(x) := \lambda^{(d-|\beta|)/2}D^\beta[G(x\sqrt{\lambda})] = \lambda^{d/2}[D^\beta G](x\sqrt{\lambda}).$$

Observe that $\int |G_{m\lambda\beta}|\,dx$ is finite and independent of λ since $|\beta| \leq 2m-1$. Also observe that the Green's function of L_λ^m, $m > 0$, is given by $\lambda^{d/2-m}G(x\sqrt{\lambda})$.

Lemma 2.1 *Let $m \in \{0,1,2,\dots\}$, $\lambda \geq 1$, and $\varepsilon \in (0,1/2)$. Let $\eta_k \in C^\infty$, $k = 1,2,3,\dots$. Assume that for any multi-index α*

$$\sup_{x\in R^d}\sum_k |D^\alpha \eta_k(x)| \leq M(\alpha).$$

Then (i) there exist constants $c_{\alpha\beta}^m$ such that for any $f \in C_0^\infty$

$$L_\lambda^m[\eta_k f] = \eta_k L_\lambda^m f + \sum_{0<|\alpha|\leq 2m-|\beta|} c_{\alpha\beta}^m \lambda^{-|\alpha|/2}[D^\alpha \eta_k](G_{m\lambda\beta} * L_\lambda^m f), \tag{2.3}$$

$$L_\lambda^m[\eta_k L_\lambda^{-m} f] = \eta_k f + \sum_{0<|\alpha|\leq 2m-|\beta|} c_{\alpha\beta}^m \lambda^{-|\alpha|/2}[D^\alpha \eta_k](G_{m\lambda\beta} * f), \tag{2.4}$$

$$L_\lambda^{-m}[\eta_k L_\lambda^m f] = \eta_k f + \sum_{0<|\alpha|\leq 2m-|\beta|} c_{\alpha\beta}^m \lambda^{-|\alpha|/2} G_{m\lambda\beta} * [(D^\alpha \eta_k) f], \tag{2.5}$$

$$L_\lambda^{-m}[\eta_k f] = \eta_k L_\lambda^{-m} f + \sum_{0<|\alpha|\le 2m-|\beta|} c^m_{\alpha\beta}\lambda^{-|\alpha|/2} G_{m\lambda\beta} * [(D^\alpha \eta_k) L_\lambda^{-m} f], \tag{2.6}$$

(ii) there exists a constant $N = N(p, d, m, \varepsilon, M)$ *such that for any* $f \in C_0^\infty$ *we have*

$$\sum_k ||L_\lambda^{\pm m/2}\{L^\varepsilon_\lambda[\eta_k f] - \eta_k L^\varepsilon_\lambda f\}||_p^p \le N\lambda^{p(\varepsilon-1/2)}||L_\lambda^{\pm m/2} f||_p^p. \tag{2.7}$$

Proof. (i) By the Leibnitz rule for certain constants $c^m_{\alpha\beta}$ we have

$$L_\lambda^m[\eta_k f] = \eta_k L_\lambda^m f + \sum_{0<|\alpha|\le 2m-|\beta|} c^m_{\alpha\beta}\lambda^{m-(|\alpha|+|\beta|)/2}[D^\alpha\eta_k]D^\beta f =:$$

$$\eta_k L_\lambda^m f + \sum_{0<|\alpha|\le 2m-|\beta|} c^m_{\alpha\beta}\lambda^{m-(|\alpha|+|\beta|)/2}[D^\alpha\eta_k]D^\beta L_\lambda^{-m}[L_\lambda^m f],$$

and this gives us (2.3). This equation is true not only for $f \in C_0^\infty$ but also for $f \in H_p^{2m}$. Therefore we can replace f by $L_\lambda^{-m} f$ in (2.3). This gives us (2.4). Taking conjugate operators to those participating in (2.3) we get (2.5). Equation (2.6) is obtained from (2.5) by substitution of $L_\lambda^{-m} f$ instead of f.

(ii) According to [KP] for a constant c_ε

$$L^\varepsilon_\lambda f = c_\varepsilon \int_0^\infty (e^{-\lambda t/2}T_t - I) f \,\frac{dt}{t^{1+\varepsilon}},$$

$$I_k(x) := L^\varepsilon_\lambda[\eta_k f](x) - \eta_k L^\varepsilon_\lambda f(x) = c_\varepsilon \int_0^\infty e^{-\lambda t/2}(T_t[\eta_k f](x) - \eta_k T_t f(x))\frac{dt}{t^{1+\varepsilon}} =$$

$$c_\varepsilon \int_0^\infty e^{-\lambda t/2}\int_{R^d}\psi(\frac{y}{\sqrt{t}})[\eta_k(x+y) - \eta_k(x)]f(x+y)\,dy\frac{dt}{t^{1+d/2+\varepsilon}},$$

where $\psi(x) = (2\pi)^{-d/2}\exp(-|x|^2/2)$. Hence by the Hölder (or by the Minkowski) inequality $|I_k| \le I_{1k}I_{2k}$ where

$$I_{1k}(x) = c_\varepsilon\{\int_0^\infty e^{-\lambda t/2}\int_{R^d}\psi(\frac{y}{\sqrt{t}})|\eta_k(x+y) - \eta_k(x)|\,dy\frac{dt}{t^{1+d/2+\varepsilon}}\}^{1-1/p},$$

$$I_{2k}(x) = \{\int_0^\infty e^{-\lambda t/2}\int_{R^d}\psi(\frac{y}{\sqrt{t}})|\eta_k(x+y) - \eta_k(x)||f(x+y)|^p\,dy\frac{dt}{t^{1+d/2+\varepsilon}}\}^{1/p}.$$

Since $\sum_k |\eta_k(x+y) - \eta_k(x)| \le N|y|$, we have

$$\int_{R^d}\psi(\frac{y}{\sqrt{t}})|\eta_k(x+y) - \eta_k(x)|\,dy \le N\int_{R^d}|y|\psi(\frac{y}{\sqrt{t}})\,dy = Nt^{(d+1)/2},$$

$$I_{1k} \le N\lambda^{(\varepsilon-1/2)(1-1/p)}, \quad \lambda^{(1/2-\varepsilon)(p-1)}\sum_k ||I_k||_p^p \le N\sum_k ||I_{2k}||_p^p =$$

$$N\int_{R^d}\int_0^\infty e^{-\lambda t/2}\int_{R^d}\psi(\frac{y}{\sqrt{t}})\sum_k|\eta_k(x+y)-\eta_k(x)||f(x+y)|^p\,\frac{dtdydx}{t^{1+d/2+\varepsilon}}\leq$$

$$N\int_0^\infty e^{-\lambda t/2}\int_{R^d}|y|\psi(\frac{y}{\sqrt{t}})\,dy\frac{dt}{t^{1+d/2+\varepsilon}}||f||_p^p=N\lambda^{\varepsilon-1/2}||f||_p^p.$$

This gives us (2.7) for $m=0$. In the general case when m is even we apply (2.3), (2.6) and the fact that the operators L_λ and $f\to G_{m\lambda\beta}*f$ commute. For instance, by (2.6)

$$L_\lambda^{-m}\{L_\lambda^\varepsilon[\eta_k f]-\eta_k L_\lambda^\varepsilon f\}=L_\lambda^\varepsilon(L_\lambda^{-m}[\eta_k L_\lambda^m g])-L_\lambda^{-m}[\eta_k L_\lambda^m(L_\lambda^\varepsilon g)]=$$

$$L_\lambda^\varepsilon[\eta_k g]-\eta_k L_\lambda^\varepsilon g+\sum_{0<|\alpha|\leq 2m-|\beta|}c_{\alpha\beta}^m\lambda^{-|\alpha|/2}G_{m\lambda\beta}*\{L_\lambda^\varepsilon[(D^\alpha\eta_k)g]-(D^\alpha\eta_k)L_\lambda^\varepsilon g\}$$

where $g=L_\lambda^{-m}f$. After this one can apply (2.7) with $m=0$ along with the Minkowski inequality (to estimate the norm of $G_{m\lambda\beta}*h$ through the norm of h), and then one gets (2.7) for the sign $+$ and for even m. The inequality with $-$ is proved similarly on the basis of (2.3).

To consider the remaining possibility of odd m, we notice that as is well known for any n

$$||L_\lambda^n f||_p\sim||L_\lambda^{n-1/2}f||_p+\sum_{i=1}^d||L_\lambda^{n-1/2}f_{x^i}||_p,$$

$$||L_\lambda^{n-1/2}f||_p\leq\lambda^{-1/2}||L_\lambda^n f||_p. \tag{2.8}$$

Hence

$$\sum_k||L_\lambda^{\pm m/2}\{L_\lambda^\varepsilon[\eta_k f]-\eta_k L_\lambda^\varepsilon f\}||_p^p\leq N\sum_k||L_\lambda^{\pm m/2-1/2}\{L_\lambda^\varepsilon[\eta_k f]-\eta_k L_\lambda^\varepsilon f\}||_p^p+$$

$$N\sum_{k,i}||L_\lambda^{\pm m/2-1/2}\{L_\lambda^\varepsilon[\eta_k f]_{x^i}-(\eta_k L_\lambda^\varepsilon f)_{x^i}\}||_p^p\leq N\lambda^{p(\varepsilon-1/2)}||L_\lambda^{\pm m/2-1/2}f||_p^p+$$

$$N\sum_{k,i}||L_\lambda^{\pm m/2-1/2}\{L_\lambda^\varepsilon[\eta_k f_{x^i}]-\eta_k L_\lambda^\varepsilon f_{x^i}\}||_p^p+$$

$$N\sum_{k,i}||L_\lambda^{\pm m/2-1/2}\{L_\lambda^\varepsilon[\eta_{kx^i}f]-\eta_{kx^i}L_\lambda^\varepsilon f\}||_p^p,$$

and (2.7) follows. The lemma is proved.

Proof of Theorem 2.1. It suffices to consider only $f\in C_0^\infty$ and since $||L_\lambda^{-n}(I-\Delta)^n||<\infty$ for any $n\in(-\infty,\infty)$, where the operator norm is taken in L_p, we only need to show that for a $\lambda=\lambda(d,n,M,\delta)\geq 1$

$$||L_\lambda^n f||_p^p\leq N\sum_k||L_\lambda^n[\zeta_k f]||_p^p,\quad\sum_k||L_\lambda^n[\zeta_k f]||_p^p\leq N||L_\lambda^n f||_p^p. \tag{2.9}$$

We consider three cases of possible values of n.
First case: $|n|\in\{0,1,2,...\}$. If $n=-m\leq 0$, we notice that

$$\sum_k || \sum_{0<|\alpha|\leq 2m-|\beta|} c^m_{\alpha\beta}\lambda^{-|\alpha|/2}G_{m\lambda\beta}*[(D^\alpha\zeta_k)L^{-m}_\lambda f]||^p_p \leq$$

$$N\lambda^{-p/2}\sum_{0<|\alpha|\leq 2m-|\beta|}\sum_k ||G_{m\lambda\beta}*[(D^\alpha\zeta_k)L^{-m}_\lambda f]||^p_p \leq$$

$$N\lambda^{-p/2}\sum_{|\alpha|\leq 2m}\sum_k ||[(D^\alpha\zeta_k)L^{-m}_\lambda f]||^p_p \leq N\lambda^{-p/2}||L^{-m}_\lambda f||^p_p.$$

Hence by (2.6) and the Minkowski inequality for an appropriate N,λ

$$(1-N\lambda^{-p/2})||L^n_\lambda f||^p_p \leq \delta^{-1}\sum_k ||\zeta_k L^{-m}_\lambda f||^p_p - N\lambda^{-p/2}||L^{-m}_\lambda f||^p_p \leq$$

$$N\sum_k ||L^{-m}_\lambda[\zeta_k f]||^p_p \leq (N+N\lambda^{-p/2})||L^{-m}_\lambda f||^p_p,$$

which yields (2.9).

If $n=m\geq 0$, it suffices to repeat the above argument using (2.3) instead of (2.6).
Second case: $2|n|\in\{1,2,3,...\}$. Here we again use (2.8). Then from the first case we get

$$||L^n_\lambda f||^p_p \leq N\sum_k ||L^{n-1/2}_\lambda[\zeta_k f]||^p_p + N\sum_{k,i}||L^{n-1/2}_\lambda[\zeta_k f_{x^i}]||^p_p \leq$$

$$N\sum_k ||L^{n-1/2}_\lambda[\zeta_k f]||^p_p + N\sum_{k,i}||L^{n-1/2}_\lambda(\zeta_k f)_{x^i}||^p_p + N\sum_{k,i}||L^{n-1/2}_\lambda[\zeta_{kx^i}f]||^p_p \leq$$

$$N\sum_k ||L^n_\lambda[\zeta_k f]||^p_p + N\lambda^{-p/2}||L^n_\lambda f||^p_p,$$

which for large λ gives the first inequality in (2.9). In like manner the second one is proved.
Third case: $n=m+\varepsilon$, $2|m|\in\{0,1,2,3,...\}$, $\varepsilon\in(0,1/2)$. To prove (2.9) in this situation it suffices to notice that by (2.7) and by the second case

$$||L^n_\lambda f||^p_p = ||L^m_\lambda[L^\varepsilon_\lambda f]||^p_p \leq N\sum_k ||L^m_\lambda[\zeta_k L^\varepsilon_\lambda f]||^p_p \leq N\sum_k ||L^m_\lambda L^\varepsilon_\lambda(\zeta_k f)||^p_p +$$

$$N\lambda^{p(\varepsilon-1/2)}||L^m_\lambda f||^p_p = N\sum_k ||L^n_\lambda(\zeta_k f)||^p_p + N\lambda^{p(\varepsilon-1/2)}||L^m_\lambda f||^p_p \leq$$

$$N\sum_k ||L^n_\lambda(\zeta_k f)||^p_p + N\lambda^{-p/2}||L^n_\lambda f||^p_p,$$

$$\sum_k ||L_\lambda^n(\zeta_k f)||_p^p = \sum_k ||L_\lambda^m[L_\lambda^\varepsilon(\zeta_k f)]||_p^p \leq$$

$$N\sum_k ||L_\lambda^m[\zeta_k L_\lambda^\varepsilon f]||_p^p + N\lambda^{-p/2}||L_\lambda^n f||_p^p \leq N||L_\lambda^n f||_p^p.$$

The theorem is proved.

Remark 2.1 In the proof of the second inequality in (2.2) we did not use the second condition in (2.1). Therefore, under the conditions of Lemma 2.1 alone the operator $f \to \{L_1^n \eta_k L_1^{-n} f : k = 1, 2, 3, ...\}$ is a bounded operator from L_p into $L_p(R^d, l_p)$. Its conjugate is also bounded, which means that under the conditions of Lemma 2.1 for any $n \in (-\infty, \infty)$, $q \in (1, \infty)$ there is a constant $N = N(n, p, d, M)$ such that for any sequence of functions $g_k \in L_q$ satisfying $\sum_k ||g_k||_q^q < \infty$ we have

$$||\sum_k \eta_k g_k||_{n,q}^q \leq N \sum_k ||g_k||_{n,q}^q. \tag{2.10}$$

Interestingly enough, this shows that, actually, the first inequality in (2.2) follows from the second one. To see this it suffices to take in (2.10) $\eta_k = \zeta_k(\sum_i \zeta_i^2)^{-1}$ and $g_k = \zeta_k f$.

References

[B] N. Burger, *Espace des fonctions à variation moyenne bornée sur un espace de nature homogène*, C. R. Acad. Sc. Paris, Vol. 286 (1978), 139-142.

[KP] M.A. Krasnoselskii et al. "Integral operators in spaces of summable functions", Nauka, Moscow, 1966 in Russian; English translation: Noordhoff International Publishing, Leyden, 1976.

[K] N.V. Krylov, *On L_p–theory of stochastic partial differential equations in the whole space*, in preparation.

[S] R.S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. and Mech., Vol. 16, no. 9 (1967), 1031-1060.

[T1] H. Triebel, "Theory of function spaces", Birkhäuser Verlag, Basel–Boston–Stuttgart, 1983.

[T2] H. Triebel, "Theory of function spaces II", Birkhäuser Verlag, Basel–Boston–Berlin, 1992.