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Remarks on AC Continuity and the Spectral Representation of Stationary SαS Sequences

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§ Introduction

Let $\mathcal{Z}$, $\mathcal{R}$ and $\mathcal{C}$ denote the set of all integers, real numbers and complex numbers, respectively, For a measure space $(\mathcal{S}, \mu)$, let $L^p(\mathcal{S}, \mu; \mathcal{X})$ denote the space of Bochner measurable p integrable functions with values in a Banach space $\mathcal{X}$. If $\mathcal{X} = \mathcal{C}$, we will simply write $L^p(\mathcal{S}, \mu)$; also if $(\mathcal{S}, \mu)$ are absent then it is assumed that $\mathcal{S} = [0, 2\pi)$ and μ is the Lebesgue measure on $[0, 2\pi)$. For a function $f \in L^1(\mathcal{X})$, $\widehat{f}$ will stand for the Fourier transform of f defined by $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(s)e^{-isn}ds, n \in \mathcal{Z}$.

Let $\{f_n : n \in \mathcal{Z}\}$ be a sequence of complex numbers. N. Wiener [14] defined the spectrum of $\{f_n\}$ under the condition

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{N} f_{n+k}\overline{f_n}\right\} \tag{1}$$

exists for each $k \in \mathcal{Z}$. Under the condition (1) (for $k = 0$) it is shown that $s(t) = \sum' \frac{f_n e^{int}}{n}$, $t \in [0, 2\pi)$ exists, where $\sum'$ denotes the sum with $n = 0$ omitted and the convergence is in L^2. However $s(t)$ in general is not of bounded variation [14] and hence the meaning of $f_n = \int_0^{2\pi} e^{-int}ds(t)$ seems to be mysterious ([12]). In fact Wiener in his subsequent work did not pursue this point. In [6] the meaning of the above spectral representation is given as a generalized function in the continuous parameter case.

The idea of a generalized spectrum for a sequence $\{X(n) : n \in \mathcal{Z}\}$ taking values in a Banach space $\mathcal{X}$ was introduces in [11] as follows

Definition 1 *Let $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$ and S_X be the linear operator from the set $\mathcal{P}$ of trigonometric polynomials defined by*

$$S_X(\varphi) = \sum \widehat{\varphi}(k)X(-k), \qquad \varphi \in \mathcal{P}.$$

We call S_X the (generalized) spectral distribution of X.

Let C and AC be the Banach spaces of complex 2π-periodic functions which are continuous and absolutely continuous on $[0, 2\pi]$ with norms

$$||f||_C = \sup_{[0,2\pi]} |f(t)| \quad \text{and} \quad ||f||_{AC} = |f(0)| + var(f)$$

where $var(f)$ denotes the variation of the function f on $[0, 2\pi]$. It is known that if S_X is continuous with respect to $||\cdot||_C$ then $X(n) = \int_0^{2\pi} e^{-int}\mu(dt)$, where μ is an $\mathcal{X}$-valued measure of finite semivariation for $\mathcal{X}$ reflexive. In other words, $X = \{X(n) : n \in \mathcal{Z}\}$ is harmonizable.

In this work we examine conditions on a sequence $X = \{X(n) : n \in \mathcal{Z}\}$ so that the Wiener type spectrum exists and we obtain conditions which guarantee the spectral representation of X(n) in terms of Wiener spectrum. We also show that mixed moving average $S\alpha S$ sequences have a continuous Wiener spectrum and have a spectral representation. We use ideas of [2].

Definition 2 *Let $\mathcal{X}$ be a reflexive separable Banach space. A sequence $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$ is said to be AC-continuous if*

$$||S_X(\varphi)|| \leq C||\varphi||_{AC}, \forall \varphi \in \mathcal{P}. \tag{2}$$

If X is AC-continuous then the extension of S_X to AC also is denoted by S_X. The following theorem gives the general form of the spectral distribution of an AC-continuous sequence.

Theorem 1 *Let $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$, a separable reflexive Banach space, be an AC-continuous sequence. Then there exists a bounded measurable function $x : [0, 2\pi) \to \mathcal{X}$ such that*

$$S_X(\varphi) = \varphi(0)X(0) - \int_0^{2\pi} \varphi'(t)x(t)dt, \ \varphi \in AC. \tag{3}$$

We refer to $x(\cdot)$ as a spectral function of X. Moreover:

(i) *$x(\cdot)$ is unique a.e. up to a constant vector,*

(ii) *the Fourier coefficients of $x(\cdot)$ are:*

$$\widehat{x}(n) = \frac{X(n) - X(0)}{2\pi in}, \ if n \neq 0 \text{ and } \widehat{x}(0) = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt \tag{4}$$

(iii) *if $\psi \in AC$, then $Y(n) = S_X(e^{in\cdot}\psi)$ is an AC-continuous sequence and its spectral function $y(\cdot)$ (as defined by (3)) is given by*

$$y(t) = \psi(t)x(t) - \int_0^t \psi'(s)x(s)ds + const. \tag{5}$$

Remark. It is easy to see that a sequence with S_X given by (3) on $\mathcal{P}$ is AC-continuous.

Proof: Since AC is isometric to $\mathcal{C} \oplus L^1$ by the mapping $AC \ni \varphi \to \varphi(0) \oplus \varphi\prime \in \mathcal{C} \oplus L^1$, $(AC)^*$ can be identified with $\mathcal{C} \oplus L^\infty$. More precisely, for every $u \in (AC)^*$ there is $a \in \mathcal{C}$ and $f_u \in L^\infty$ such that

$$u(\varphi) = a\varphi(0) + \int_0^{2\pi} \varphi\prime(t) f_u(t)dt, \quad \varphi \in AC.$$

Therefore for every $b^* \in M(X)^*$, where $M(X) = \overline{sp}\{X(n) : n \in \mathcal{Z}\}$ there is $a_{b^*} \in C$ and $f_{b^*} \in L^\infty$ such that

$$(S_X^* b^*)(\varphi) = \varphi(0) a_{b^*} + \int_0^{2\pi} \varphi\prime(t) f_{b^*}(t)dt, \quad \varphi \in AC.$$

The mapping $M(X)^* \ni b^* \to f_{b^*} \in L^\infty$ is linear and bounded, therefore from the lifting theorem ([10], 48, 52) there is a bounded measurable function $f : [0, 2\pi) \to M(X)^{**} = M(X)$ such that $f_{b^*}(t) = b^*(f(t))$, dt-a.e. for each $b^* \in M(X)^*$. The mapping $M(X)^* \ni b^* \to a_{b^*}$ is a continuous functional, so it has the form $a_{b^*} = b^*(z)$, for some $z \in M(X)$. Summing up, for all $\varphi \in AC$ and $b^* \in M(X)^*$

$$\begin{aligned} (S_X^* b^*)(\varphi) &= \varphi(0) b^*(z) + \int_0^{2\pi} \varphi\prime(t) b^*(f(t))dt \\ &= b^*(\varphi(0)z + \int_0^{2\pi} \varphi\prime(t) f(t)dt) \end{aligned}$$

Thus

$$S_X(\varphi) = \varphi(0)z + \int_0^{2\pi} \varphi\prime(t) f(t)dt, \quad \varphi \in AC.$$

Setting $\varphi \equiv 1$ we conclude that $z = S_X(1) = X(0)$. Setting $x(t) = -f(t)$ we obtain (3). In the "moreover" part the properties (i) and (ii) are trivial. To prove (iii) first note that $S_Y(\varphi) = S_X(\varphi\psi)$ and, because AC is a Banach algebra, $\|S_Y(\varphi)\| \leq C\|\psi\|_{AC} = C_1\|\varphi\|_{AC}$. Therefore $\{Y(n) : n \in \mathcal{Z}\}$ is AC-continuous. Let $y(t) = \psi(t)x(t) - \int_0^t \psi\prime(s)x(s)ds$. Then for every $\varphi \in AC$

$$\begin{aligned} &\varphi(0)Y(0) - \int_0^{2\pi} \varphi\prime(t) y(t)dt \\ &= \varphi(0)S_X(\psi) - \int_0^{2\pi} \varphi\prime(t)\psi(t)x(t)dt + \int\int 1_{[0,t]}(s)\varphi\prime(t)\psi(s)x(s)dsdt \\ &= \varphi(0)S_X(\psi) - \int_0^{2\pi} \varphi\prime(t)\psi(t)x(t)dt + \int \psi\prime(s)x(s)(\varphi(2\pi) - \varphi(s))ds \\ &= \varphi(0)S_X(\psi) + \varphi(2\pi)\int_0^{2\pi} \psi\prime(s)x(s)ds - \int_0^{2\pi} (\psi\varphi)\prime(t)x(t)dt \end{aligned}$$

$$= \varphi(0)[S_X(\psi) - X(0)\varphi(0) - S_X(\psi)] - \int_0^{2\pi} (\psi\varphi)\prime(t)x(t)dt$$

Hence $S_X(\varphi\psi) = S_Y(\psi)$. Thus (4) follows then from (i). □

Remark. An AC-continuous sequences does not have to be norm bounded. Let $\mathcal{X} = \mathcal{C}$, the complex numbers, and consider $X(n) = S_X(e^{in\cdot})$, $n \neq 0$, and $X(0) = 0$, where $S_X(\varphi) = -\int \varphi\prime(t)x(t)dt$ with

$$x(t) = \sum_{\nu=1}^{\infty} \nu^{-\beta} e^{i\nu^\alpha} e^{i\nu(t+\pi)}$$

$0 \leq t \leq 2\pi$. Then by ([15], vol.1, p. 200) with $1 - \frac{\alpha}{2} < \beta < 1$, we get that $x(t)$ is continuous and by (4) above, for $k \neq 0$, $X(k) = -[2\pi ik(k^{-\beta} \exp(ik^\alpha) \exp(i\pi k))]$. Thus $X = \{X(k) : k \in \mathcal{Z}\}$ is not bounded but is AC-continuous.

In the next theorem we relate a spectral function $x(\cdot)$ of X to the Wiener spectrum $s(\cdot)$. Let $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$ be a sequence in a separable Banach space $\mathcal{X}$ and N be a nonnegative integer. Then for $t \in [0, 2\pi)$:

$$S_N(X;t) = \sum_{n=-N}^{N} \frac{X(n)}{n} e^{int}, \qquad \Sigma_N(X;t) = \frac{1}{N+1} \sum_{k=1}^{N} S_k(X;t)$$

As usual, if f is an integrable $\mathcal{X}$-valued function then

$$s_N(f;t) = \sum_{n=-N}^{N} \widehat{f}(n) e^{int}, \qquad \sigma_N(f;t) = \frac{1}{N+1} \sum_{k=0}^{N} s_k(f;t)$$

Theorem 2 *Let $\mathcal{X}$ be a separable reflexive Banach space and $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$. Then the following are equivalent:*

(i) *X is AC-continuous,*

(ii) $\|\Sigma_N(X;t)\| \leq M < \infty$ *a.e. for all N.*

In either case there exists a bounded function $s(t)$ such that $\Sigma_N(X;t) \to s(t)$ weakly e.a. Moreover, if $x(\cdot)$ is a spectral function of X then

$$x(t) = \frac{itX(0) + s(t)}{2\pi i} + const. \quad a.e. \tag{6}$$

Proof: $(i) \Rightarrow (ii)$. By Theorem 1 there is a bounded function x such that $\widehat{x}(n) = \frac{i}{2\pi}(\frac{X(0)}{n} - \frac{X(n)}{n})$, $n \neq 0$. Because $x(\cdot)$ is defined up to a constant, we may assume that $\widehat{x}(0) = 0$. Therefore

$$s_N(x;t) = \frac{i}{2\pi} \left\{ \sum_{n=-N}^{N} \frac{e^{int}}{n} X(0) - \sum_{n=-N}^{N} \frac{X(n)}{n} e^{int} \right\}$$

and so $\sigma_N(x;t) = \frac{i}{2\pi}(X(0)\sigma_N(\omega;t) - \Sigma_N(X;t))$, where $\omega(t) = i(\pi - t)$, $0 < t < 2\pi$ and $\omega(0) = 0$. Since $x(\cdot)$ is bounded, $\|\sigma_N(x;t)\| \leq M < \infty$ a.e. for all N and that for every $x^* \in \mathcal{X}^*$, $< \sigma_N(x;t), x^* >$ converges a.e. to

$< x(t), x^* >$ (Thms III 3.4 and III 3.9 in [15]). Since $\sigma_N(\omega;t)$ converges boundedly to $\omega(t)$ for every t we conclude (ii) and the weak convergence of $\Sigma_N(X;t)$ to $s(t) = X(0)w(t) + 2\pi i x(t)$. From the last equation we obtain (6).

$(ii) \Rightarrow (i)$ Observe that $\Sigma_N(X;t) = S_N(X;t) - \frac{1}{N+1}\sum_{n=-N}^{N} sgn(n)X(n)e^{int}$. Let $\varphi(t) = \sum_{k=-N}^{N} \widehat{\varphi}(k)\exp ikt,\ t \in [0,2\pi)$. Then for $n \geq N$

$$S_X(\varphi) = \sum_{k=-n}^{n} \widehat{\varphi}(k)X(-k) = \widehat{\varphi}(0)X(0) + \frac{1}{2\pi}\int_0^{2\pi} \varphi(t) \sum_{k=-n}^{n} e^{-ikt}X(-k)\ dt$$

Integrating by parts we get

$$S_X(\varphi) = \widehat{\varphi}(0)X(0) + \frac{i}{2\pi}\int_0^{2\pi} \varphi\prime(t)\left[\sum_{k=-n}^{n} \frac{e^{ikt}X(k)}{k}\right] dt.$$

The integral on the RHS above equals

$$\int_0^{2\pi} \varphi\prime(t)\left[\sum_{k=-n}^{n} \frac{e^{ikt}X(k)}{k}\left(1 - \frac{|k|}{n+1}\right)\right] dt +$$
$$\frac{1}{n+1}\int_0^{2\pi} \varphi\prime(t)\left[\sum_{k=-n}^{n} sgn(k)e^{ikt}X(k)\right] dt.$$

Using the fact that in the second integrand $\widehat{\varphi}(k) = 0,\ |k| > N$, we obtain

$$\begin{aligned} S_X(\varphi) &= \widehat{\varphi}(0)X(0) - \frac{i}{2\pi}\int_0^{2\pi} \varphi\prime(t)\Sigma_n(X,-t)dt \\ &+ \frac{i}{2\pi(n+1)}\int_0^{2\pi} \varphi\prime(t)\left[\sum_{k=-N}^{N} sgn(k)e^{-ikt}X(-k)\right] dt\ . \end{aligned}$$

Thus $\|S_X(\varphi)\| = |\widehat{\varphi}(0)|\|X(0)\| + \frac{M}{2\pi}\|\varphi\|_{AC} + \frac{2N}{2\pi(n+1)}\|\varphi\|_{AC}\sup_{|k|<N}\|X(k)\|$. Since $|\widehat{\varphi}(0)| \leq (2\pi)\|\varphi\|_{AC}$ and last term goes to zero as $n \to \infty$ we get $\|S_X(\varphi)\| \leq 2\pi(\|X(0)\| + \frac{M}{2\pi})\|\varphi\|_{AC}$. $\square$

Remark. Note that if X is bounded then from $\Sigma_N(X;t) = S_N(X;t) - \frac{1}{N+1}\sum_{n=-N}^{N} sgn(n)X(n)e^{int}$ it follows that $\sup_N \|\Sigma_N(X;t)\| \leq M < \infty$ a.e. iff $\sup_N \|S_N(X;t)\| \leq M' < \infty$ a.e.

As a corollary from Theorems 1 and 2 we obtain that for a given sequence $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$, if $\sup_N \|\Sigma_N(X;t)\| \leq M < \infty$ a.e. (or X is bounded and $\sup_N \|S_N(X;t)\| \leq M' < \infty$ a.e.) then $S_N(X;t)$ is (C,1) summable a.e. to a bounded function s(t) and the sequence X has the representation

$$X(n) = X(0) - \int_0^{2\pi} \frac{d}{dt}(e^{-int})x(t)dt, \qquad n \in \mathcal{Z} \tag{7}$$

with $x(t) = \frac{itX(0)+s(t)}{2\pi i} + const$. If the function $x(\cdot)$ has a cadlag (continuous from right and having limits from left) version, the formula (7) can by written in terms of an integral w.r.t. x in the sense of [7]. Recall that if $x(t)$ is a function on $[0, 2\pi]$ (both endpoints are included) with values in $\mathcal{X}$ and f is a complex function on $[0, 2\pi]$ with $var(f) < \infty$, then the integral

$$\int_{[0,2\pi]}^{\oplus} f(t)dx(t) \stackrel{df}{=} \lim_{\Pi} \left\{ \sum_{k=1}^{n} f(t_k)[x(t_k) - x(t_{k-1})] \right\}$$

exists, where the limit is over the directed set of partitions $\Pi = \{0 = t_0 < t_1 < ... < t_n = 2\pi\}$ of $[0, 2\pi]$. The following properties of the integral above are proved in [7].

Proposition 1 ([7]). *Suppose that x is cadlag.*

(i) *If f is absolutely continuous then*

$$\int_{[0,2\pi]}^{\oplus} f(t)dx(t) = f(2\pi)x(2\pi) - f(0)x(0) - \int_{0}^{2\pi} x(t)df(t)$$

(ii) *If $\{g_\alpha\ ,\ \alpha \in \mathcal{A}\}$, is a net of functions with $\sup_\alpha(var(g_\alpha)) < \infty$, which converges pointwise to g then*

$$\int_{[0,2\pi]}^{\oplus} g_\alpha(t)dx(t) \xrightarrow{\alpha} \int_{[0,2\pi]}^{\oplus} g(t)dx(t).$$

Sequences with spectral function admitting a cadlag version are discussed in the next theorem. In the sequel we will say that a function f has the property (F) if for every $t \in [0, 2\pi]$, the limits $f(t+)$ and $f(t-)$ exists and $f(t+) - f(t-) = 2f(t),\quad t \in [0, 2\pi)$, where $0- = 2\pi-$. Note that a function having property (F) is bounded.

Theorem 3 *Let $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$ be a sequence in a separable reflexive Banach space. The following conditions are equivalent.*

(i) $||\Sigma_N(X;t)|| \leq M < \infty$ *a.e. for all N and $s(t) = *-\lim \Sigma_N(X;t)$ (which exists a.e. by Theorem 2) has the property (F).*

(ii) *For all $t \in [0, 2\pi)$, $\Sigma_N(X;t)$ converges (in norm) to a function having the property (F).*

(iii) *For all $t \in [0, 2\pi)$, $\Sigma_N(X;t)$ converges weakly to a function having the property (F).*

(iv) *There is a bounded cadlag function $x(t)$, $t \in [0, 2\pi]$ such that $x(0) = 0$, $x(2\pi) = X(0)$ and*

$$X(n) = \int_{[0,2\pi]}^{\oplus} e^{-int}dx(t),\ n \in \mathcal{Z}. \tag{8}$$

If, in addition, X is bounded then the conditions (i)-(iv) are equivalent to

(v) *For all* $t \in [0, 2\pi)$, $S_N(X;t)$ *converges to a function having the property (F).*

Proof: (i) $\Rightarrow$ (ii). From Theorem 2 it follows that X is AC continuous and $s(t) = X(0)w(t) + 2\pi i x(t)$, where $x(\cdot)$ is a spectral function of X with $\widehat{x}(0) = 0$. By Theorem 1 (ii), $\widehat{s}(n) = \frac{X(n)}{n}$, $n \neq 0$ and $\widehat{s}(0) = 0$. Therefore $\sum \frac{X(n)}{n} e^{int}$ is a Fourier series of $s(\cdot)$and consequently $\Sigma_N(X;t) = \sigma_N(s;t)$. Since s has the property (F), by repeating the arguments used in [15], III 2.21 and 3.4, we conclude (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Denote $s(t) = *\text{-}\lim \Sigma_N(X;t)$, $t \in [0, 2\pi)$. From [1] , (2.1), it follows that for every $x^* \in \mathcal{X}^*$, $\Sigma_N(x^*(X);t) = \sigma_N(x^*(s);t)$. Therefore by the properties of the Fejer kernel for every $N \in \mathcal{Z}$

$$|x^*(\Sigma_N(X;t))| = |\sigma_N(x^*(s);t)| \leq \|x^*\| \sup_u \|s(u)\|$$

Hence $\|\Sigma_N(X;t)\| \leq M < \infty$, $N \in \mathcal{Z}$. By Theorem 2, X is AC continuous and $\tilde{x}(t) = \frac{s(t) - X(0)\omega(t)}{2\pi i}$ is its spectral function, where $\omega(t) = i(\pi - t), 0 < t < 2\pi$ and $\omega(0) = 0$. Observe that $\tilde{x}$ has (F) property and

$$\tilde{x}(t+) - \tilde{x}(t-) = \frac{s(t+) - s(t-)}{2\pi i}, \quad t \in (0, 2\pi)$$
$$\tilde{x}(0+) - \tilde{x}(2\pi-) = \frac{s(0+) - s(2\pi-)}{2\pi i} - X(0)$$
$$\tilde{x}(0+) = \frac{s(0+) - i\pi X(0)}{2\pi i}$$

Therefore if for $t \in (0, 2\pi)$ we set

$$\begin{aligned} x(t) &= \frac{s(t) - X(0)\omega(t)}{2\pi i} - \frac{s(0+) - i\pi X(0)}{2\pi i} + \frac{s(t+) - s(t-)}{4\pi i} \\ &= \frac{itX(0) + s(t) - s(0)}{2\pi i} + \frac{[s(t+) - s(t-)] - [s(0+) - s(2\pi-)]}{4\pi i}, \end{aligned}$$

and define $x(0) = 0$, $x(2\pi) = X(0)$, then x satisfies the conditions of (iv) Moreover, by Proposition 1 (i)

$$\int_{[0,2\pi]}^{\oplus} e^{-int} dx(t) = x(2\pi) - x(0) - \int_0^{2\pi} x(t) d(e^{-int}) = X(n), \ n \in \mathcal{Z}.$$

Observe also that $x(2\pi) - x(2\pi-) = \frac{s(0+) - s(2\pi-)}{2\pi i}$.

(iv) $\Rightarrow$ (i). Suppose (iv). Then, by Proposition 1 (i), for every trigonometric polynomial φ

$$\begin{aligned} S_X(\varphi) &= \sum \widehat{\varphi}(k) X(-k) = \int_{[0,2\pi]}^{\oplus} \varphi(t) dx(t) \\ &= \varphi(0)X(0) - \int_0^{2\pi} \varphi'(t) x(t) dt. \end{aligned}$$

Since x is bounded, $||S_X(\varphi)|| \leq C||\varphi||_{AC}$. Therefore X is AC continuous, and x is a spectral function of X. Since x has a version with property (F), from Theorem 2 and the relation between s and x established there we conclude (i).

Now assume that X is bounded and that (ii) holds. Let $s(t) = \lim \Sigma_N(X;t)$, $t \in [0, 2\pi)$. Then, as we already have noticed, $\Sigma_N(X;t) = \sigma_N(s;t)$, and $S_N(X;t) = s_N(s;t)$. Since $||\widehat{s}(n)|| \leq C/|n|$, $n \neq 0$, from Hardy's tauberian theorem ([15] , III, 1.26) we obtain (v). The converse implication is obvious from ([15] , III, 1.3). □

Below there are three examples of sequences which admit the representation (8). In the first we show that for a harmonizable sequence the function $x(t)$ is equal $\mu(0,t]$, as is expected. In the next two we discuss $S\alpha S$ stochastic sequences. We now derive representations from Theorem 3 in specific examples.

Example 1. Harmonizable Sequences

A sequence $X = \{X(n) : n \in \mathcal{Z}\} \subseteq \mathcal{X}$ is called harmonizable if there exists an $\mathcal{X}$-valued measure μ on $[0, 2\pi)$ such that $X(n) = \int_0^{2\pi} e^{-int}\mu(dt)$, $n \in \mathcal{Z}$. If X is harmonizable then a simple computation shows that for $t \in (0, 2\pi)$

$$\begin{aligned} s(t) &= \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{X(n)}{n} e^{int} = \lim \int_0^{2\pi} \sum_{n=-N}^{N} \frac{e^{ik(t-s)}}{n} \mu(ds) \\ &= i\left\{ \int_0^{2\pi} (\pi - t + s) 1_{[0,t)}(s)\mu(ds) + \int_0^{2\pi} (s - t - \pi) 1_{(t,2\pi)}(s)\mu(ds) \right\} \\ &= i\left\{ \int_0^{2\pi} s\mu(ds) - t\mu\{t\} - tX(0) + t\mu\{t\} + \pi\mu[0,t) - \pi\mu(t,2\pi) \right\}. \end{aligned}$$

and $s(0) = i\int_0^{2\pi}(s - \pi)1_{(0,2\pi)}(s)\mu(ds)$. Therefore s has the property (F) and for $t \in [0, 2\pi)$

(i) $itX(0) + s(t) - s(0) = i\pi\{\mu\{0\} + 2\mu(0,t) + \mu\{t\}\}$,

(ii) $s(t+) - s(t-) = 2\pi i\mu\{t\}$, (here $0- = 2\pi-$).

Hence

$$x(t) = \frac{itX(0) + s(t) - s(0)}{2\pi i} + \frac{[s(t+) - s(t-)] - [s(0+) - s(2\pi-)]}{4\pi i} = \mu(0,t]$$

and $x(2\pi) = \mu[0, 2\pi)$.

Example 2. Mixed Moving Average $S\alpha S$ Sequences

A *stochastic sequence* is a collection $X = \{X(n) : n \in \mathcal{Z}\}$ of complex random variables. A stochastic sequence X is *stationary* (SP) if for any $N \in \mathcal{N}$ and any $n_1, .., n_N \in \mathcal{Z}$, the random vectors $< X(n_1), ..., X(n_N) >$ and $< X(n_1+1), ..., X(n_N+1) >$ have the same distributions. A stochastic sequence $X = \{X(n) : n \in \mathcal{Z}\}$ is *symmetric α stable* ($S\alpha S$), $0 < \alpha < 2$, if for each $N \in \mathcal{N}$, $\mathbf{n} = \{n_1, ... n_N\} \in \mathcal{Z}^N$ and $\lambda = \{\lambda_1, ..., \lambda_N\} \in \mathcal{C}^N$, the random variable $X(\lambda, \mathbf{n}) = \sum_{p=1}^{N} \lambda_p X(n_p)$ is radially $S\alpha S$, i.e., its characteristic function $\varphi(z) = Eexp[iRe(\overline{z}X(\lambda, \mathbf{n}))]$, $z \in \mathcal{C}$, has the form $\varphi(z) = exp(-c(\lambda, \mathbf{n})|z|^\alpha)$, $z \in \mathcal{C}$, where $c(\lambda, \mathbf{n})$ is a nonnegative constant. If $X = \{X(n) : n \in \mathcal{Z}\}$ is an $S\alpha S$ sequence with $1 < \alpha \leq 2$ then in the space $\mathcal{M}(X)$ of linear combinations of $X(n)$, $n \in \mathcal{Z}$, the function $\| \sum_{p=1}^{N} \lambda_p X(n_p) \|_\alpha$ defined by $\{c(\lambda, \mathbf{n})\}^{1/\alpha}$, for $1 < \alpha < 2$,and $\{E| \sum_{p=1}^{N} \lambda_p X(n_p)|^2\}^{1/2}$, for $\alpha = 2$, is a norm equivalent to convergence in probability and equal up to a constant to $L^p(\Omega)$ norm for $1 < p < \alpha$. For $S\alpha S$ sequences with $\alpha < 2$ stationarity is equivalent to the condition that for any $N \in \mathcal{N}$, $t, n_1, ..., n_N \in \mathcal{Z}$ and $\lambda_1, ..., \lambda_N \in \mathcal{C}$, $\| \sum_{k=1}^{N} X(n_k)\lambda_k \|_\alpha = \| \sum_{k=1}^{N} X(n_k + 1)\lambda_k \|_\alpha$. In other words an $S\alpha S$ sequence ($\alpha < 2$) is stationary if there is an isometry $U : \overline{\mathcal{M}(X)} = M(X) \overset{onto}{\to} M(X)$ such that $U^n X(m) = X(m+n)$, $m, n \in \mathcal{Z}$. If $\alpha = 2$ the latter condition leads to larger than stationary class of processes called **wide sense** stationary. Following [13] an $S\alpha S$ process is called a *mixed moving average* (MMA) if it has the form

$$X(n) = \int_{X \times \mathcal{Z}} f(x, n-m) M(dx, dm)$$

where M is independently scattered $S\alpha S$ random measure on $X \times \mathcal{Z}$ with control measure $Q \times l$ (i.e. $Eexp[iRe(\overline{z}M(A,B))] = exp(-Q(A)l(B))|z|^\alpha)$, $z \in \mathcal{C}$), Q is a σ-finite measure on X, l is counting measure and f is a complex measurable function in $L^\alpha(X \times \mathcal{Z}, Q \times l)$. A *moving average* (MA) $S\alpha S$ process is a process of the form

$$X(n) = \sum_{n=-\infty}^{\infty} f(n-m) M\{m\}$$

where M is independently scattered $S\alpha S$ random measure on $\mathcal{Z}$ with control measure l and $f \in L^\alpha(\mathcal{Z}, l)$. MA processes form a subclass (proper if $\alpha < 2$) of MMA processes. Clearly any MMA process is stationary for $\alpha < 2$ (and wide sense stationary if $\alpha = 2$) since

$$E\exp[iRe(\overline{z}X(\lambda, \mathbf{n}))] = exp\left\{ -|z|^\alpha \sum_{m=-\infty}^{\infty} \int_X |\sum_{p=1}^{N} \lambda_p f(x, n_i - m)|^\alpha Q(dx) \right\},$$

$N \in \mathcal{N}$, $n_1, ..., n_N \in \mathcal{Z}$ and $\lambda_1, ..., \lambda_N, z \in \mathcal{C}$.

Theorem 4 *Let $X = \{X(n) : n \in \mathcal{Z}\}$ be an MMA sequence, $1 < \alpha < 2$. Then there is a continuous in probability $S\alpha S$ stable process $x(t)$, $t \in [0, 2\pi]$ such that for every $n \in \mathcal{Z}$,*

$$X(n) = \int_{[0,2\pi]}^{\oplus} \exp(-int) dx(t)$$

One can take $x(t) = \frac{itX(0) + s(t) - s(0)}{2\pi i}$, $0 \leq t < 2\pi$ and $x(2\pi) = X(0)$, where $s(t) = \lim_{N \to \infty} \left\{ \sum_{n=-N}^{N} \frac{X(n)}{n} e^{int} \right\}$.

Proof: The proof is a slight modification of the argument from [3], p.456. Denote $\mathcal{X} = L^\alpha(X,Q)$. Let for $f \in L^\alpha(\mathcal{Z}, l; \mathcal{X})$, $H_N(f)(\cdot) = \sum_{n=-N}^{N} \frac{f(\cdot - n)}{\pi n}$ and $H(f)(\cdot) = \lim_{N\to\infty} H_N(f)(\cdot)$ in $L^\alpha(\mathcal{Z}, l; \mathcal{X})$, if exists. Since $\mathcal{X}$ is an UMD space ([2]) H is a bounded operator from $L^\alpha(\mathcal{Z}, l; \mathcal{X})$ to $L^\alpha(\mathcal{Z}, l; \mathcal{X})$ and $\sup_N \|H_N\| = K < \infty$ (see [2] , 2.8 and 2.18). $H(f)$ will be referred to as a *Hilbert transform* of f.

If $X = \{X(n) : n \in \mathcal{Z}\}$ is an MMA sequence, $1 < \alpha < 2$, then by definition the mapping $\Phi : X(n) \to f_n \in L^\alpha(\mathcal{Z}, l; L^\alpha(X,Q))$, where $\{f_n(k)\}(x) = f(x, n-k) = \{f_0(k-n)\}(x)$ is an isometry from $M(X)$ into $L^\alpha(\mathcal{Z}, l; L^\alpha(X,Q))$. Observe that

$$\begin{aligned} e^{is\cdot} H_N(e^{-is\cdot} f_0)(\cdot) &= e^{is\cdot} \sum_{n=-N}^{N} \frac{f_0(\cdot - n)\exp(-is(\cdot - n))}{\pi n} \\ &= \sum_{n=-N}^{N} \frac{f_0(\cdot - n)e^{isn}}{\pi n} = \sum_{n=-N}^{N} \frac{f_n(\cdot)e^{isn}}{\pi n}, \end{aligned}$$

and because $\mathcal{X}$ is an UMD space the latter converges in $L^\alpha(\mathcal{Z}, l; \mathcal{X})$ to $R(s) = e^{is\cdot} H(e^{-is\cdot} f_0)(\cdot)$ for every $s \in [0, 2\pi]$. Applying the mapping Φ^{-1} and using the above mentioned properties of the Hilbert transform we see that:

(i) $S_N(X;t) = \pi\Phi^{-1}(e^{it\cdot} H_N(e^{-it\cdot} f_0)(\cdot)) \xrightarrow{N} \pi\Phi^{-1}\left(e^{it\cdot} H(e^{-it\cdot} f_0)\right) \stackrel{df}{=} s(t)$,

(ii) $\sup \|S_N(X;t)\| = \pi \sup \|e^{it\cdot} H_N(e^{-it\cdot} f_0)(\cdot))\|$
$\leq \pi K \|e^{-it\cdot} f_0\| = \pi K \|f_0\| < \infty$,

(iii) $s(t) = \pi\Phi^{-1}\left(e^{it\cdot} H(e^{-it\cdot} f_0)\right)$ is continuous function of t, because for any fixed $g \in L^\alpha(\mathcal{Z}, l; \mathcal{X})$ the function $[0, 2\pi] \ni t \to e^{it\cdot} g(\cdot) \in L^\alpha(\mathcal{Z}, l; \mathcal{X})$ is continuous from $[0, 2\pi]$ to $L^\alpha(\mathcal{Z}, l; \mathcal{X})$; also clearly $s(0) = s(2\pi)$.

Therefore from Theorem 3 it follows that $X(n)$ has the desired representation and, by (ii), $x(\cdot)$ is continuous. Note that, by definition of $x(t)$ and $s(t)$, the random variable $x(t)$ is a limit of linear combinations of $X(n)$'s, hence $\{x(t) : 0 \leq t \leq 2\pi\}$ is an $S\alpha S$ process. □

Remark. It was proved in [11] that if X is MA then x must not be a measure. For continuous parameter MA processes a representation similar to the above was obtained in [5] under additional assumption that the function f belongs also to $L^1(\mathcal{R})$. In fact, the spectral representation for groups generated by measure-preserving transformations obtained by Fife [9] and then generalized in the series of papers by Berkson, Gillespie *et.al.* [1], [2], [3] to the case of power bounded groups of operators on UMD spaces, yields the existence of a cadlag type spectral function $x(t)$ for an arbitrary stationary process (discrete or continuous time) in $L^p(\Omega, P)$, $1 < p < \infty$). We state here only a version for $S\alpha S$ stationary sequences and prove it using arguments in [4]

Theorem 5 ([9], Theorem 1, [2], Theorems 3.1 and 5.16). *Let $X = \{X(n) : n \in \mathcal{Z}\}$ be an $S\alpha S$ stationary sequence, $1 < \alpha < 2$. Then:*

(i) *For each $t \in [0, 2\pi)$, $\lim \left(\sum_{n=-N}^{N} \frac{X(n)}{n} \exp(int)\right) = s(t)$ exists in $M(X)$.*

(ii) *For each $t \in [0, 2\pi)$, $\lim \left(N^{-1} \sum_{n=0}^{N-1} X(n) \exp(int)\right) = p(t)$ exists in $M(X)$.*

(iii) *The process*

$$x(t) \stackrel{df}{=} \frac{itX(0) + s(t) - s(0)}{2\pi i} + \frac{p(t) - p(0)}{2}, \quad 0 < t < 2\pi,$$

and $x(2\pi) = X(0)$, is a bounded, cadlag $S\alpha S$ process on $[0, 2\pi)$.

(iv) *For every $n \in \mathcal{Z}$, $X(n) = \int_{[0,2\pi]}^{\oplus} e^{-int} dx(t)$.*

Proof: The proof is a reconstruction of an argument used in [9]. Consider the probability space $(\Omega, \mathcal{F}, P_X)$, where $\Omega = \mathcal{C}^Z, \mathcal{F}$ is the cylindrical σ - algebra and P_X is the distribution of the sequence X. Then $\{X(n), n \in \mathcal{Z}\}$, and the sequence of random variables on Ω defined by $\tilde{X}(n)(\omega) = \omega(n)$ have the same finite dimensional distributions and from now on they will be identified. Let for a measurable function $f : \Omega \to \mathcal{C}$, $T^n f = f \circ \phi^n$, $n \in \mathcal{Z}$, where $\phi(\omega)(\cdot) = \omega(\cdot + 1)$, $\omega \in \Omega$. Then ϕ is a measure preserving point transformation and T^n is a group of isometries on $L^p(\Omega, \mathcal{F}, P_X)$ for all $1 \leq p < \infty$. Let $1 < p < \alpha$. For every $0 \leq t < 2\pi$ consider two sequences of operators

$$H_N(t) = \sum_{n=-N}^{N} \frac{T^n f}{n} e^{int} \quad \text{and}$$

$$P_N(t) = N^{-1} \sum_{n=0}^{N-1} T^n f e^{int},$$

$N = 1, 2, \ldots$. From [4], it follows that for every $0 \leq t < 2\pi$ the strong limits $H(t) = \lim_N H_N(t)$ and $P(t) = \lim_N P_N(t)$ exists a.e. and in $L^p(\Omega, \mathcal{F}, P_X)$ and both $\sup_t \|H(t)\|_p$ and $\sup_t \|P(t)\|_p$ are finite. The symbol $\|S\|_p$ stands here for the norm of a linear operator $S : L^p(\Omega, \mathcal{F}, P_X) \to L^p(\Omega, \mathcal{F}, P_X)$. On the other hand the group T^n restricted to $L^2(\Omega, \mathcal{F}, P_X)$ is unitary, and denoting by $E(dt)$ its spectral measure, the argument used in Example 1 shows that for $f \in L^2(\Omega, \mathcal{F}, P_X)$

$H(t)f$ has the property (F) as a function into $L^2(\Omega, \mathcal{F}, P_X)$,

$[H(t+) - H(t-)]f = 2\pi i E\{t\} f = 2\pi i P(t), \quad t \in [0, 2\pi).$

Since $L^2(\Omega, \mathcal{F}, P_X)$ is dense in $L^p(\Omega, \mathcal{F}, P_X)$ and the functions $\|H(t)\|_p$ and $\|P(t)\|_p$ are bounded, $H(t)f = \lim H_N(t)f$ has (F) property as a function from $[0,2\pi)$ to L^p and $[H(t+)-H(t-)]f = 2\pi i P(t)f, \quad t \in [0,2\pi)$, for every $f \in L^p(\Omega, \mathcal{F}, P_X)$. Therefore, by Theorem 3,

$$T^n f = \int_{[0,2\pi]}^{\oplus} e^{-int} dE(t)f, \quad n \in \mathcal{Z},\ f \in L^p(\Omega, \mathcal{F}, P_X),$$

where $E(t)f = (2\pi i)^{-1}(itf + H(t)f - H(0)f) + 2^{-1}(P(t) - P(0)),\ t \in [0,2\pi)$ and $E(2\pi) = I$. Note that $E(0) = 0$ and, from uniform boundedness of $H(\cdot)$ and $P(\cdot)$, $\sup_t \|E(t)\|_p < \infty$ (in fact, the argument used in Example 1 shows that $E(t)$ is a continuous extension of $E(0,t]$ to $L^p(\Omega, \mathcal{F}, P_X)$). If we now take $f(\omega) = \omega(0)$, $\omega \in \Omega$, then $T^n f = X(n)$ and the theorem is proved. The process $x(t) = E(t)f$ is evidently $S\alpha S$ because its linear combinations are limits of linear combinations of $X(n)$'s. □

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