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On Asymptotic Behavior of Solutions of the Wave Equations Perturbed by a Fast Markov Process

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Abstract

We study the asymptotic behavior of the distributions of solutions of the randomly perturbed wave equations $\partial^2 u_\epsilon(t,x)/\partial t^2 = a^2(y(t/\epsilon))\Delta u_\epsilon(t,x)$ in a bounded region G with a smooth boundary Γ with some initial and zero boundary conditions on Γ, where y(t) is an ergodic jump Markov process. It is shown that the distributions of the random field $u_\epsilon(t/\epsilon, x)$ coincide asymptotically with the distribution of a random field which is represented by its series expansion in terms of the eigenfunctions of the Laplace operator Δ in the region G with zero boundary condition on Γ with coefficients depending on a sequence of independent Wiener processes.

1 Introduction

Let $(Y, \mathcal{C})$ be a measurable space and $y(t)$ be a homogeneous jump Markov process in $(Y, \mathcal{C})$ with transition probability $P(t, y, C)$. We consider the wave equation of the form

$$\frac{\partial^2 u_\epsilon(t,x)}{\partial t^2} = a^2(y(\frac{t}{\epsilon}))\Delta u_\epsilon(t,x),\ x \in G \tag{1.1}$$

with the initial and boundary conditions:

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$$u_\epsilon(0,x) = f(x),\ \frac{\partial}{\partial t} u_\epsilon(t,x)|_{t=0} = g(x),\ u_\epsilon(t,x)|_{x\in\Gamma} = 0, \tag{1.2}$$

where $G \subset R^d$ is a bounded region with a smooth boundary Γ.

We suppose $y(t)$ is an ergodic process with the ergodic distribution $\rho(dy)$. Let $\bar{a} > 0$ be defined by

$$\bar{a}^2 = \int a^2(y)\rho(dy). \tag{1.3}$$

Then it follows from [5] that $u_\epsilon(t,x)$ converges in an appropriate metric to the solution of the averaged equation

$$\frac{\partial^2 \bar{u}(t,x)}{\partial t^2} = \bar{a}^2 \Delta \bar{u}(t,x) \tag{1.4}$$

with the same initial and boundary conditions (1.2).

It is well-known that the solution to (1.4) has the form

$$\bar{u}(t,x) = \sum_{n=1}^{\infty} r_n \psi_n(x) \cos(\bar{a}\sqrt{\lambda_n} t + \varphi_n), \tag{1.5}$$

where the functions $\psi_n(x)$ are determined by the eigenvalue system

$$\left.\begin{array}{c} \Delta\psi_n(x) + \lambda_n \psi_n(x) = 0, \quad \psi_n(x) = 0,\ x \in \Gamma, \\ \int_G \psi_n(x)\psi_m(x)dx = 0,\ n \neq m \text{ and } \int_G \psi_n^2(x)dx = 1, \\ 0 < \lambda_1 < \lambda_2 < \ldots \end{array}\right\} \tag{1.6}$$

The constants r_n and φ_n are determined by the initial functions f and g as follows: Let $A_n = \int_G \psi_n(x) f(x)dx$, $\bar{a}\sqrt{\lambda_n} B_n = \int_G \psi_n(x) g(x)dx$. Let $\sum_{n=1}^{\infty}(|A_n| + |B_n|) < \infty$. Then $r_n = (A_n^2 + B_n^2)^{1/2}$, $r_n \cos\ \varphi_n = A_n$, $r_n \sin\varphi_n = -B_n$.

In this paper we present an asymptotic representation for $u_\epsilon(\frac{t}{\epsilon}, x)$, as $\epsilon \to 0$. We use the notions of weak convergence and asymptotic coincidence of distributions. Let us recall these concepts here. Let $V_\epsilon(t,x)$, $V_0(t,x)$, $\hat{V}_\epsilon(t,x)$ be random fields. $V_\epsilon(t,x)$ converges weakly to $V_0(t,x)$ if $E\phi(V_\epsilon(t_1,x_1),\ldots,V_\epsilon(t_n,x_n)) \to E\phi(V_0(t_1,x_1),\ldots,V_0(t_n,x_n))$ and the distributions of $V_\epsilon(t,x)$ and $\hat{V}_\epsilon(t,x)$ coincide asymptotically if

$$E\{\phi(V_\epsilon(t_1,x_1),...,V_\epsilon(t_n,x_n) - \phi(\hat{V}_\epsilon(t_1,x_1),..,\hat{V}_\epsilon(t_n,x_n)\} \to 0$$

as $\epsilon \to 0$ for all bounded continuous function $\phi : R^n \to R$ and $t_1,\ldots,t_n \in R_+$; $x_1,\ldots,x_n \in R^d$.

Let us make some historical remark relevent to this work. The problems considered here are special cases of a general problem of investigation of dynamic systems under the influence of random perturbations. The study of these problems began by N.M. Krylov and N.N. Bogolubov in the 1930s in connection with some problems of mathematical physics. The work was extended by a student of N.N. Bogolubov, namely I.I. Gikhman, who for these purposes laid down the foundation of the theory of stochastic differential equations in 1940s. Major contribution in the development of asymptotic methods in investigation of randomly perturbed dynamical systems was made by R. Khasminskii in his well known monograph (1969). G. Papanicolaou, D. Stroock and S. Varadhan (1977) proposed a very general martingale method for studying the asymptotic behavior of continuous and jump dynamical systems, in particular they found a diffusion approximation for such systems for large time. We consider here randomly perturbed wave equations, where the corresponding dynamical system is of infinite dimension. Infinite dimensional dynamical systems under various assumptions were studied by G. Papanicolaou and S. Varadhan (1973), G. Papanicolaou (1978). We note that the problems under consideration are closely connected with some problems of stochastic P.D.E. which were developed by N.V. Krylov and B.L. Rozovskii (1977), (1979), (1982), E. Pardoux (1977), B.L. Rozovskii (1985). Randomly perturbed PDE's were considered by R. Figare, E. Orlandi and G. Papanicolaou (1982); R. Khasminskii, F. Hoppensteadt, H. Salehi (1994); F. Hoppensteadt, H. Salehi, A. Skorokhod (1994).

The main results of this paper which consists of two theorems and several corollaries are stated in Section 2; their proofs are provided in Section 3. In Section 4 applications to vibration of string and membrain are discussed.

2 Results

This section consists of three parts. In part 1 our assumption are stated. Part 2 covers some auxilary materials and part 3 is devoted to the statements of the main results.

2.1 Assumptions

(2.1) We assume that the process $y(t)$ has some regularity properties.

Let $\mathrm{Var}(m_1 - m_2)$ denote the variation of the difference of two measures. We assume

$$\sup_{y\epsilon Y} \int_0^\infty \mathrm{Var}(P(t,y,\cdot) - \rho(\cdot))dt < \infty. \tag{A$_1$}$$

This condition implies the asymptotic normality, as $T \to \infty$, of the integrals

$\int\limits_0^T f(y(t))dt$ for every bounded measurable function f.

(2.2) Other assumptions are concerned with the region G and the initial functions $f(x)$ and $g(x)$. Let $\psi_k(x)$ and λ_k satisfy the relations

$$\Delta\psi_k(x) + \lambda_k\psi_k(x) = 0 \text{ in } G,\ \ \psi_k(x)|_{x\in\Gamma} = 0,\ \int_G \psi_k^2(x)dx = 1.$$

We suppose that

$$\left.\begin{array}{ll} 1) & \{\psi_k(x)\} \text{ is an orthogonal basis in } L_2(G) \\ 2) & \sup\limits_{k,x\in G} |\psi_k(x)| < \infty, \psi_k(x) \text{ are continuous} \\ 3) & \sum \lambda_k^{-1} < \infty \\ 4) & f(x) = \sum\limits_{n=1}^{\infty} A_n\psi_n(x), g(x) = \sum\limits_{n=1}^{\infty} B_n\psi_n(x) \end{array}\right\} \qquad (A_2)$$

and these series are uniformly convergent.

2.2 Auxiliary Results

The proofs of our main results are based on some general asymptotic theorems for randomly perturbed ordinary differential equations with almost periodic coefficients.

Let $x_\epsilon(t)$ be the solution of the equation

$$\frac{dx_\epsilon(t)}{dt} = a(t, x_\epsilon(t),\, y(\tfrac{t}{\epsilon})),\; x_\epsilon(0) = x_0 \tag{2.3}$$

where $a(t,x,y) : R_+ \times R^d \times Y \to R^d$.

We assume the existence of the derivatives $a'_x(t,x,y)$ and $a''_{xx}(t,x,y)$ and that the function $a(t,x,y)$, $a'_x(t,x,y)$, $a''_{xx}(t,x,y)$ are bounded measurable in y and continuous in t,x uniformly with respect to y. Besides we suppose $\int a(t,x,y)\rho(dy) = 0$.

Suppose that the transition probability $P(t,y,C)$ of the process $y(t)$ satisfies Assumption (A_1). Let

$$\left.\begin{array}{ll} R(y,C) = & \int_0^\infty (P(t,y,C) - \rho(C))dt, \\ G_t\varphi(x) = & \int\int \left(\dfrac{\partial}{\partial x}(\varphi'(x), a(t,x,y')), a(t,x,y)\right)\rho(dy)R(y,dy'). \end{array}\right\} \tag{2.4}$$

Theorem 2.1. *Let for any φ in $C^{(2)}(R^d)$ the limit*

$$\hat{G}\varphi(x) = \lim(1/2T)\int_{t-T}^{t+T} G_u\varphi(x)du, \tag{2.5}$$

exist as $T \to \infty$ uniformly in $t > T$.
Then the process $\tilde{x}_\epsilon(t) = x_\epsilon(\frac{t}{\epsilon})$ converges weakly to a diffusion process $\hat{x}(t)$ with the generator $\hat{G}$ and the initial value $\hat{x}(0) = x_0$.

2.3 Main Results

Using the Fourier method we obtain the following representation for the solution of the stochastically perturbed equation (1.1):

$$u_\epsilon(t,x) = \sum r_k\psi_k(x)\exp\{\sqrt{\lambda_k}\int_0^t K(y\tfrac{s}{\epsilon}))\sin 2\bar{a}_k(\theta_k(\epsilon,s)+s)ds\} \times \cos\bar{a}_k(\theta_k(\epsilon,t)+t), \tag{2.6}$$

where $\bar{a}_k = \sqrt{\lambda_k}\bar{a}$, and θ_k's are the solutions of the differential equations

$$\frac{d\theta_k(\epsilon,t)}{dt} = \sqrt{\lambda_k}K(y(\tfrac{t}{\epsilon}))(1+\cos 2\bar{a}_k(\theta_k(\epsilon,t)), \quad \theta_k(\epsilon,0)=\varphi_k. \tag{2.7}$$

The coefficients r_k's and φ_k's are determined by the initial functions $f(x)$ and $g(x)$, and the kernel $K(y) = (a^2(y)-\bar{a}^2)/2\bar{a}$.

For the asymptotic behavior of the summands in (2.6), we introduce the random functions

$$Z_0(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))ds,$$
$$Z_k(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))(\cos 2\bar{a}_k\theta_k(\epsilon,s)+s)ds, \ k=1,2,\ldots$$
$$V_k(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))\sin 2\bar{a}_k(\theta_k(\epsilon,s)+s)ds, \ k=1,2,\ldots$$

Theorem 2.2 *Let the transition probability $P(t,y,C)$ of the process $y(t)$ satisfy Assumption (A_1). Then for all m, the functions*

$$(Z_0(\epsilon,\tfrac{t}{\epsilon}),\ Z_1(\epsilon,\tfrac{t}{\epsilon}),\ldots,Z_m(\epsilon,\tfrac{t}{\epsilon}),\ V_1(\epsilon,\tfrac{t}{\epsilon}),\ldots,V_m(\epsilon,\tfrac{t}{\epsilon}))$$

converges weakly, as $\epsilon\to 0$, to the system of real-valued independent Wiener processes

$$(Z_0(t),\ Z_1(t),\ldots,Z_m(t),V_1(t),\ldots,V_m(t))$$

for which $EZ_k(t) = EV_k(t) = 0$ and

$$EZ_0^2(t) = 2ct,\ EZ_k^2(t) = EV_k^2(t) = ct,$$
$$c = \int_0^\infty \int\int K(y)K(y')(P(t,y,dy')-\rho(dy'))\rho(dy)dt.$$

Corollary 2.1. *Let Assumption (A_2) be satisfied, and let*

$$\sum r_k e^{\lambda_k t} < \infty \text{ for all } \quad t>0.$$

Then the distributions of $u_\epsilon(\frac{t}{\epsilon},x)$ coincide asymptotically with the distributions of the random field

$$\hat{u}_\epsilon(t,x) = \sum_{k=1}^\infty r_k\psi_k(x)e^{\sqrt{\lambda_k}V_k(t)}\cos[\bar{a}_k(\frac{t}{\epsilon}+Z_0(t)+\sqrt{\lambda_k}Z_k(t))+\varphi_k].$$

By equation (1.5), the solution to the averaged equation (1.4) is given by the formula

$$\bar{u}(t,x) = \sum_{n=1}^\infty r_n\psi_n(x)\cos(\bar{a}\sqrt{\lambda_n}t+\varphi_n). \tag{*}$$

By Corollary 2.1, the distributions of $u_\epsilon(t/\epsilon, x)$, $u_\epsilon(t,x)$ being the solution of equations (1.1)-(1.2), coincide asymptotically with the distributions of the random field $\hat{u}_\epsilon(\epsilon t, x)$ given by

$$\hat{u}_\epsilon(\epsilon t, x) = \sum_{n=1}^{\infty} r_n(\epsilon t)\psi_n(x)\cos(\bar{a}\sqrt{\lambda_n}t + \varphi_n(\epsilon t)), \qquad (**)$$

where

$$r_n(t) = r_n \exp\{\sqrt{\lambda_n}V_n(t)\},$$

$$\varphi_n(t) = \varphi_n + \bar{a}\sqrt{\lambda_n}\{Z_0(t) + \sqrt{\lambda_n}Z_n(t)\}.$$

We therefore obtain:

Corollary 2.2. *Formulas (*) and (**) have the same form, but in formula (**) the amplitudes and the phases of eigenoscillations $r_n(\epsilon t)$ and $\varphi_n(\epsilon t)$ are random and slowly change with the change of t.*

Corollary 2.3. *If t changes in a bounded interval then the expressions in the right hand sides of (*) and (**) coinside asymptotically, since $r_n(\epsilon t) \to r_n$ and $\varphi_n(\epsilon t) \to \varphi_n$, as $\epsilon \to 0$.*

Note that the joint distribution of the processes

$$Z_0(\epsilon t), \dots, Z_m(\epsilon t), \dots, V_1(\epsilon t), \dots, V_m(\epsilon t), \dots$$

coincides with the joint distribution of the processes

$$\sqrt{\epsilon}Z_0(t), \dots, \sqrt{\epsilon}Z_m(t), \dots, \sqrt{\epsilon}V_1(t), \dots, \sqrt{\epsilon}V_m(t), \dots$$

Therefore we have:

Corollary 2.4. *The distributions of the random function $\hat{u}_\epsilon(\epsilon t, x)$ in (**) coincides with the distributions of the random function*

$$u^*_\epsilon(t,x) =$$

$$\sum_{n=1}^{\infty} r_n \exp\{\sqrt{\epsilon}\sqrt{\lambda_n}V_n(t)\}\psi_n(x)\cos\{\bar{a}\sqrt{\lambda_n}t + \varphi_n + \sqrt{\epsilon}(\varphi_n(t) - \varphi_n)\}\ .$$

It is easy to see that as $\epsilon \to 0$ we have

$$\frac{1}{\sqrt{\epsilon}}(u^*_\epsilon(t,x) - \bar{u}(t,x)) \to \hat{u}_0(t,x),$$

where $\widehat{u}_0(t,x)$ is given by

$$\widehat{u}_0(t,x) =$$

$$\sum_{n=1}^{\infty} r_n\psi_n(x)\{\sqrt{\lambda_n}V_n(t)\cos(\bar{a}\sqrt{\lambda_n}t+\varphi_n) - (\varphi_n(t)-\varphi_n)\sin(\bar{a}\sqrt{\lambda_n}t+\varphi_n)\}.$$

Thus:

Corollary 2.5: $\frac{1}{\sqrt{\epsilon}}(u_\epsilon(t,x) - \bar{u}(t,x))$ *converges weakly to* $\hat{u}_0(t,x)$.

3 Proofs

We proceed to make some transformations of the problem. Let $\psi_k(x)$ and λ_k be determined by the relation (1.6).
Let

$$\alpha_n(\epsilon,t) = \int_G u_\epsilon(t,x)\psi_n(x)dx,\ n = 1,2,\dots \tag{3.1}$$

Then $\alpha_n(\epsilon,t)$ is the solution of the ordinary differential equation

$$\frac{d^2}{dt^2}\alpha_n(\epsilon,t) + \lambda_n a^2(y(\tfrac{t}{\epsilon}))\alpha_n(\epsilon,t) = 0 \tag{3.2}$$

with the initial conditions

$$\left.\begin{aligned} \alpha_n(\epsilon,0) = \alpha_n^0 &= \int_G f(x)\psi_n(x)dx, \\ \frac{d}{dt}\alpha_n(\epsilon,t)|_{t=0} &= \alpha_n^1 = \int_G g(x)\psi_n(x)dx. \end{aligned}\right\} \tag{3.3}$$

We introduce new variables $r_n(\epsilon,t)$ and $\varphi_n(\epsilon,t)$ which are related to $\alpha_n(\epsilon,t)$ by the relations

$$\alpha_n(\epsilon,t) = r_n(\epsilon,t)\cos\bar{a}_n\varphi_n(\epsilon,t) \tag{3.4}$$

$$\frac{d}{dt}\alpha_n(\epsilon,t) = -\bar{a}_n r_n(\epsilon,t)\sin\bar{a}_n\varphi_n(\epsilon,t), \tag{3.5}$$

where $\bar{a}_n = \sqrt{\lambda_n}\bar{a}$.

Lemma 3.1. $\varphi_n(\epsilon,t)$ *is the solution of the equation*

$$\frac{d\varphi_n(\epsilon,t)}{dt} = 1 + K(y(\tfrac{t}{\epsilon}))(1 + \cos\ 2\bar{a}_n\varphi_n(\epsilon,t)) \tag{3.6}$$

with the initial condition $\varphi_n(\epsilon,0) = \varphi_n$ *and*

$$r_n(\epsilon,t) = r_n\exp\{\bar{a}_n\int_0^t K(y(\tfrac{s}{\epsilon}))\sin(2\bar{a}_n\varphi_n(\epsilon,s))ds, \tag{3.7}$$

where

$$K(y) = \frac{a^2(y) - \bar{a}^2}{2\bar{a}^2}, \text{and} \tag{3.8}$$

$$\alpha_n^0 = r_n\cos\bar{a}_n\varphi_n, \quad \alpha_n^1 = r_n\sin\bar{a}_n\varphi_n. \tag{3.9}$$

Proof. We consider the derivatives of the right hand sides of the relations (3.4) and (3.5) and the relation (3.2). We can obtain the equalities:

$$\frac{d}{dt} r_n(\epsilon,t)\cos(\bar{a}_n\varphi_n(\epsilon,t)) - \bar{a}_n r_n(\epsilon,t)\frac{d}{dt}\varphi_n(\epsilon,t)\sin(\bar{a}_n\varphi_n(\epsilon,t))$$

$$= -\bar{a}_n r_n(\epsilon,t)\sin(\bar{a}_n\varphi_n(\epsilon,t)), \text{ and}$$

$$-\bar{a}_n\frac{d}{dt} r_n(\epsilon,t)\sin(\bar{a}_n\varphi_n(\epsilon,t)) - \bar{a}_n^2 r_n(\epsilon,t)\frac{d}{dt}\varphi_n(\epsilon,t)\cos(\bar{a}_n\varphi_n(\epsilon,t)$$

$$= -\lambda_n a^2(y(\frac{t}{\epsilon}))r_n(\epsilon,t)\cos(\bar{a}_n\varphi_n(\epsilon,t)).$$

These relations imply the validity of the system of the first order equations:

$$\bar{a}_n^2\frac{d\varphi_n(\epsilon,t)}{dt} = \lambda_n a^2(y(\tfrac{t}{\epsilon}))\cos^2(\bar{a}_n\varphi_n(\epsilon,t)) + \bar{a}_n^2\sin^2(\bar{a}_n\varphi_n(\epsilon,t)), \quad (3.10)$$

$$\bar{a}_n\frac{d}{dt}r_n(\epsilon,t) = \quad (3.11)$$
$$(\lambda_n a^2(y(\tfrac{t}{\epsilon})) - \bar{a}_n^2)r_n(\epsilon,t)\sin(\bar{a}_n\varphi_n(\epsilon,t))\cos(\bar{a}_n\varphi_n(\epsilon,t)).$$

(3.6) follows from (3.10); (3.11) is a linear equation with respect to $r_n(\epsilon,t)$ and (3.7) is the representation of its solution; (3.9) follows from (3.4) and (3.5). ∎

We introduce the additional variables:

$$Z_0(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))ds, \quad (3.12)$$

$$Z_n(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))\cos 2\bar{a}_n(s+\theta_n(\epsilon,s))ds,\ n=1,2,... \quad (3.13)$$

$$V_n(\epsilon,t) = \int_0^t K(y(\tfrac{s}{\epsilon}))\sin 2\bar{a}_n(s+\theta_n(\epsilon,s))ds,\ n=1,2,... \quad (3.14)$$

$$\theta_n(\epsilon,t) = \varphi_n(\epsilon,t) - t,\ n=1,2,... \quad (3.15)$$

Then we have

$$\varphi_n(\epsilon,t) = t + Z_0(\epsilon,t) + Z_n(\epsilon,t) + \varphi_n, \quad (3.16)$$

$$r_n(\epsilon,t) = r_n\exp\{\bar{a}_n V_n(\epsilon,t)\}. \quad (3.17)$$

Lemma 3.2. *Let condition (A_2) be fulfilled and $\sum r_n < \infty$. Then the function $u_\epsilon(t,x)$ may be represented in the form*

$$u_\epsilon(t,x) = \quad (3.18)$$

$$\sum_{n=1}^{\infty} r_n\psi_n(x)\exp\{\bar{a}_n V_n(\epsilon,t)\}\cos\bar{a}_n(\varphi_n + t + Z_0(\epsilon,t) + Z_n(\epsilon,t)).$$

Proof. Suppose the series in the right hand side of (3.18) is absolutely and uniformly convergent and denote its sum by $u^*(t,x)$. Then $\int u_\epsilon(t,x)\psi_n(x)dx = \int u^*(t,x)\psi_n(x)ds$, therefore $u_\epsilon(t,x) = u^*(t,x)$. Note that $y(t)$ is a step function and for fixed $\epsilon > 0$ and $T > 0$ we can divide the interval $[0,T]$ into a finite number of intervals on which $y(t)$ is a constant. It is easy to check that $\bar{a}_n V_n(\epsilon,t)$ is bounded in n and t when $y(t)$ is constant. ∎

Let n be fixed. We consider the R^{3n+1}-valued function

$$Z^{(n)}(\epsilon,t) = \tag{3.19}$$

$$(Z_0(\epsilon,t),...,Z_n(\epsilon,t),V_1(\epsilon,t),...,V_n(\epsilon,t),\theta_1(\epsilon,t),...,\theta_n(\epsilon,t)).$$

This function satisfies the equation

$$\frac{d}{dt}Z^{(n)}(\epsilon,t) = K(y(\tfrac{t}{\epsilon}))\phi^{(n)}(t,Z^{(n)}(\epsilon,t)) \tag{3.20}$$

where the coordinates of $\phi^{(n)}(t,x) : R_+ \times R^{3n+1} \to R^{3n+1}$ are determined by the relations

$$\varphi_0(t,z_0,\ldots,z_n,v_1,\ldots,v_n,\theta_1,\ldots,\theta_n) = 1$$
$$\varphi_i(,z_0,\ldots,z_n,v_1,\ldots,v_n,\theta_1,\ldots,\theta_n) = \cos 2\bar{a}_i(t+\theta_i),\ 1 \le i \le n,$$
$$\varphi_{n+i}(t,z_0,\ldots,\theta_n) = \sin 2\bar{a}_i(t+\theta_i),\ 1 \le i \le n,$$
$$\varphi_{2n+i}(t,z_0,\ldots,\theta_n) = 1 + \cos 2\bar{a}_i(t+\theta_i),\ 1 \le i \le n,$$

here

$$\phi^{(n)}(t,z) = (\varphi_0(t,z_0,\ldots,\theta_n),\ldots,\varphi_{3n}(t,z_0,\ldots,\theta_n)),$$
$$z = (z_0,\ldots,z_n,v_1,\ldots,v_n,\theta_1,\ldots,\theta_n).$$

Investigation of asymptotic behavior of the solution to equation (3.20) will be based on some general statements regarding the weak convergence of stochastic processes to a diffusion process.

We consider a family of stochastic processes $\{x_\epsilon(t),\ \epsilon \ge 0\}$ with values in R^d which are defined on R_+ and a diffusion process $x(t)$ with values in R^d with a generator G which is determined on the functions $f \in C^2(R^d)$ by the formula

$$Gf(x) = (a(x), f'(x)) + \frac{1}{2} tr\ f''(x)B(x). \tag{3.21}$$

Here $(\cdot,\cdot)$ is the scalar product in R^d, $a(x)$ is a smooth R^d-valued function on R^d, $B(x)$ is a $L(R^d)$-valued smooth function on R^d, $f'(x)$ and $f''(x)$ are the derivatives of $f(x)$; $f'(x)$ is a R^d-valued function and $f''(x)$ is a function with values in $L(R^d)$; for A in $L(R^d)$ trA denotes the trace of A.

Suppose that the process $x_\epsilon(t)$ is adapted to a filtration $\{\mathcal{F}_t^\epsilon\}$.

Proposition 3.1. *Let $x_\epsilon(0) = x_0$ for all $\epsilon > 0$, and let for all $f \in C^3(R^d)$ and $0 < t_1 < t_2 < \infty$*

$$\lim_{\epsilon \to 0} E|E\{f(x_\epsilon(t_2)) - f(x_\epsilon(t_1)) - \int_{t_1}^{t_2} Gf(x_\epsilon(s))ds \;/\mathcal{F}_{t_1}^\epsilon\}| = 0. \quad (3.22)$$

Then $x_\epsilon(t)$ converges weakly to $x(t)$, as $\epsilon \to 0$. (see [15], p. 77).

We will apply this proposition to prove Theorem 2.1.

Proof of Theorem 2.1. Denote by $(\mathcal{F}_t^\epsilon)$ the filtration which corresponds to the process $y(\frac{t}{\epsilon})$.

1) Let $g(t, x, y) : R_+ \times R^d \times Y \to R$ be a bounded function measurable in y and continuous in t, x for which the derivatives $g'_x(t, x, y)$ and $g''_{xx}(t, x, y)$ satisfy the same conditions. Then for $0 < t_1 < t_2$

$$\begin{aligned} &E\{\textstyle\int_{t_1}^{t_2} g(t, x_\epsilon(t), y(\frac{t}{\epsilon})(t))dt/\mathcal{F}_{t_1}^\epsilon\} \quad (3.23)\\ &= E\{\textstyle\int_{t_1}^{t_2} \int g(t, x_\epsilon(t), y)\rho(dy)dt + \epsilon \int_{t_1}^{t_2} Rg'_x(t, x_\epsilon(t))dt/\mathcal{F}_{t_1}^\epsilon\}\\ &\quad +O(\epsilon) + (t_2 - t_1)o(\epsilon), \end{aligned}$$

where

$$Rg'_x(t, x) = \int \int (g'_x(t, x, y'), a(t, x, y))\rho(dy)R(y, dy'). \quad (3.24)$$

Let

$$R(t, y, C) = P(t, y, C) - \rho(C), \; \bar{g}(t, x, y) = g(t, x, y) - \int g(t, x, y')\rho(dy').$$

Then formula (3.23) follows from relations:

(a)
$$\begin{aligned} &E\{\textstyle\int_{t_1}^{t_2} \bar{g}(t, x_\epsilon(t), y(\frac{t}{\epsilon}))dt/\mathcal{F}_{t_1}^\epsilon\}\\ &= E\{\textstyle\int_{t_1}^{t_2} \bar{g}(t, x_\epsilon(t_1), y(\frac{t}{\epsilon}))dt\\ &\quad + \textstyle\int_{t_1}^{t_2} \int_{t_1}^{t} (\bar{g}'_x(t, x_\epsilon(s), y(\frac{t}{\epsilon})),\; a(s, x_\epsilon(s), y(\frac{s}{\epsilon})))dsdt/\mathcal{F}_{t_1}^\epsilon\}\\ &= \epsilon E\{\textstyle\int_0^{\frac{t_2-t_1}{\epsilon}} \int \bar{g}(t_1 + \epsilon u, x_\epsilon(t_1)y')R(u, y(\frac{t_1}{\epsilon}), dy')du\\ &\quad + \textstyle\int_{t_1}^{t_2} \int_0^{\frac{t-t_1}{\epsilon}} \int (\bar{g}'_x(s + \epsilon u, x_\epsilon(s), y'),\; a(s, x_\epsilon(s), y(\frac{s}{\epsilon}))\times\\ &\qquad R(u, y(\tfrac{s}{\epsilon}), dy')duds/\mathcal{F}_{t_1}^\epsilon\}\\ &= \epsilon E\{\textstyle\int_{t_1}^{t_2} \int (g'_x(s + \epsilon u, x_\epsilon(s), y'),\; a(s, x_\epsilon(s), y(\frac{s}{\epsilon}))R(y(\frac{s}{\epsilon}), dy')ds/\mathcal{F}_{t_1}^\epsilon)\\ &\quad +O(\epsilon) + (t_2 - t_1)o(\epsilon). \end{aligned}$$

(b)
$$\begin{aligned} &E\{\textstyle\int_{t_1}^{t_2} g(t, x_\epsilon(t), y(\frac{t}{\epsilon}))dt/\mathcal{F}_{t_1}^\epsilon\}\\ &= E\{\textstyle\int_{t_1}^{t_2} \int g(t, x_\epsilon(t), y)\rho(dy)dt \;/\mathcal{F}_{t_1}^\epsilon\} + O(\epsilon)(1 + (t_2 - t_1)). \end{aligned}$$

2) Let $f(t, x) : R_+ \times R^d \to R$ be a bounded continuous function with

bounded continuous derivatives $f'_x(t,x)$, $f''_{xx}(t,x)$. Let $S_T f(t,x) = (1/2T)\int_{t-T}^{t+T} f(s,x)ds$, $t - T \geq 0$.
Then for $T \leq t_1 < t_2$,

$$E\{\int_{t_1}^{t_2} f(t,x_\epsilon(t))dt - \int_{t_1}^{t_2} S_T f(t,x_\epsilon(t))dt/\mathcal{F}_{t_1}^\epsilon\} = O(\epsilon(T+1)(t_2-t_1)) + O(T). \tag{3.25}$$

This follows from representing the left hand side of (3.25) in the form

$$\begin{aligned}
&E\{\tfrac{1}{2T}\int_{t_1+T}^{t_2-T}\int_{t-T}^{t+T}(f(t,x_\epsilon(t)) - f(t,x_\epsilon(s)))dsdt/\mathcal{F}_{t_1}^\epsilon\} + O(T)\\
&= E\{\tfrac{1}{2T}\int_{t_1+T}^{t_2-T}\int_{t-T}^{t+T}\int_s^t (f'_x(t,x_\epsilon(u)), a(u,x_\epsilon(u), y(\tfrac{u}{\epsilon}))dudsdt/\mathcal{F}_{t_1}^\epsilon\} + O(T)\\
&= O(\epsilon T(t_2-t_1)) + O(\epsilon(t_2-t_1)) + O(T),
\end{aligned}$$

where we have applied relation (3.23) to the integral $\int_s^t (\quad)$ du.
Let $\varphi \epsilon C^{(3)}(R^d)$. Then

$$\begin{aligned}
&E\{\varphi(x_\epsilon(\tfrac{t_2}{\epsilon})) - \varphi(x_\epsilon(\tfrac{t_1}{\epsilon}))/\mathcal{F}_{t_1/\epsilon}^\epsilon\}\\
&= E\{\int_{t_1/\epsilon}^{t_2/\epsilon}(\varphi'(x_\epsilon(s)),\ a(s,x_\epsilon(s), y(\tfrac{s}{\epsilon})))ds/\mathcal{F}_{t_1/\epsilon}^\epsilon\}\\
&= \epsilon\ E\{\int_{t_1/\epsilon}^{t_2/\epsilon} G_s\varphi(x_\epsilon(s))ds/\mathcal{F}_{t_1/\epsilon}^\epsilon\} + O(\epsilon) + (t_2-t_1)\tfrac{o(\epsilon)}{\epsilon}\\
&= \epsilon\ E\{\int_{t_1/\epsilon}^{t_2/\epsilon} S_T G_s\varphi(x_\epsilon(s))ds/\mathcal{F}_{t_1/\epsilon}^\epsilon\}\\
&\quad + O(\epsilon) + (t_2-t_1)\tfrac{o(\epsilon)}{\epsilon} + O(\epsilon^2(T+1)\tfrac{t_2-t_1}{\epsilon} + O(\epsilon T)\\
&= E\{\int_{t_1}^{t_2}\hat{G}\varphi(\tilde{x}_\epsilon(s))ds/\mathcal{F}_{t_1/\epsilon}^\epsilon\} + (t_2-t_1)O(\|\hat{G}\varphi - S_T G_s\varphi\|)\\
&\quad + O(\epsilon T) + (t_2-t_1)(\tfrac{o(\epsilon)}{\epsilon} + O(\epsilon T) + O(\epsilon)),
\end{aligned}$$

where we have used (3.23) in asserting the second equality and (3.25) in the verification of the third equality.
Let $\epsilon \to 0$ and $T \to \infty$ such that $\epsilon T \to 0$. Then we can apply Proposition 3.1 to complete the proof. ∎

Remark 3.1. Using the relation $((\varphi', a), a) = (\varphi'' a, a) + (\varphi', a'a)$, the operator $G_t f(x)$ may be rewritten in the form:

$$G_t f(x) = (\tilde{a}(t,x),\ f'(x)) + \frac{1}{2}tr\tilde{B}(t,x)f''(x) \tag{3.26}$$

where

$$\tilde{a}(t,x) = \int\int a'_x(t,x,y')a(t,x,y)\rho(dy)R(y,dy') \tag{3.27}$$

is a R^d-valued function and $\tilde{B}(t,x)$ is a $L(R^d)$-valued function for which

$$(\tilde{B}(t,x)z_1, z_2) = 2\int\int (a(t,x,y), z_1)(a(t,x,y'), z_2)\rho(dy)R(y,dy'). \quad (3.28)$$

Therefore

$$\int \hat{G}f(x) = (\hat{a}(x), f'(x)) + \tfrac{1}{2} tr\, \hat{B}(x),$$

where

$$\hat{a}(x) = \lim_{T\to 0} S_T \tilde{a}(t,x),\ \hat{B}(x) = \lim_{T\to\infty} S_T \tilde{B}(t,x); \quad (3.30)$$

and the existence of these limits is the main condition of the theorem.

We now present the proofs of Theorem 2.2 and Corollary 2.1. Proofs of Corollaries 2.2 — 2.5 were indicated at their enunications.

Proof of Theorem 2.2. the proof is based on an application of Theorem 2.1 to equation (3.20). In this case $d = 3n+1$ and the elements of the matrix $\tilde{B}(t,z)$, which we denote by $\tilde{b}_{ij}(t,z)$, are determined by the equalities:

$$\tilde{b}_{ij}(t,z) = 2c\varphi_i(t,z)\varphi_j(t,z)$$

(here $z = (z_0, \ldots, z_n, v_1, \ldots, v_n, \theta_1, \ldots, \theta_n)$) and elements $\hat{b}_{ij}(z)$ of the matrix $\hat{B}(z)$ are determined by equalities $\hat{b}_{ij}(z) = 2c \lim_{T\to\infty} S_T\{2\varphi_i(t,z)\varphi_j(t,z)\}$. Therefore

$$\hat{b}_{ij}(z) = \begin{cases} 0 & \text{if} \quad 0 \le i \ \ne j \le 2, \\ 2c & \text{if} \quad i = 0,\ j = 0, \\ c & \text{if} \quad 1 \le i = j \le 2n. \end{cases}$$

The elements (coordinates) of the vector $\tilde{a}(t,z) = (\tilde{a}_0(t,z), ..., a_{3n}(t,z))$ are determined by the relations

$$\tilde{a}_i(t,z) = c\{(\sum_{k=0}^{n} \frac{\partial}{\partial z_k}\varphi_i(t,z)\varphi_k(t,z) + \sum_{k=1}^{n} \frac{\partial}{\partial v_k}\varphi_i(t,z)\varphi_{k+n}(t,z)$$
$$+ \sum_{k=1}^{n} \frac{\partial}{\partial \theta_k}\varphi_i(t,z)\varphi_{k+2n}(t,z))\}.$$

We can see that

$$\hat{a}_i(z) = \lim_{T\to\infty} S_T\{\tilde{a}_i(t,z)\} = 0, \quad 0 \le i < 3n.$$

We note that $\theta_k(\epsilon, t) = Z_0(\epsilon,t) + Z_k(\epsilon,t)$, $1 \le k \le n$. Thus the equation (3.20) satisfies the conditions of Theorem 2.1, and hence the process $(Z_0(\epsilon, \frac{t}{\epsilon}), \ldots, Z_n(\epsilon, \frac{t}{\epsilon}), V_1(\epsilon, \frac{t}{\epsilon}), \ldots, V_n(\epsilon, \frac{t}{\epsilon}))$ converges weakly to the diffusion process $(\hat{Z}_0(t), \ldots, \hat{Z}_1(t), \hat{V}_1(t), \ldots, \hat{V}_n(t))$ whose drift term is zero and whose diffusion matrix is diagonal with the most left top corner diagonal term being 2c and the remaining diagonal elements are c. ∎

Proof of the Corollary 1. It is easy to see that the statement of Corollary 2.1 follows from formula (3.18) and Theorem 2.2 if the series in the right hand side of (3.18) converges uniformly in ϵ for ϵ small enough. As $\psi_n(x)$ are bounded so the convergence needed becomes a consequence of the following lemma.

Lemma 3.3. *We have*

$$E\ \exp\{\bar{a}_n V_n(\epsilon,t)\} \le b_1 \exp\{\epsilon \bar{a}_n^2 b_2 t\}, \tag{3.31}$$

where $b_i > 0$, $1 \le i \le 2$, *are constants.*

Proof. Let $H_n(\epsilon,t) = \exp\{\bar{a}_n V_n(\epsilon,t)\}$. Then

$$H_n(\epsilon,t) = 1 + \bar{a}_n \int_0^t K(y(\tfrac{s}{\epsilon}))\cos 2\bar{a}_n(s+\theta_n(\epsilon,s))ds +$$
$$+\bar{a}_n^2 \iint_{0<u<s<t} H_n(\epsilon,u)K(y(\tfrac{u}{\epsilon}))K(y(\tfrac{s}{\epsilon}))\cos\ 2\bar{a}_n\ \cos\ 2\bar{a}_n(\theta_n(\epsilon,s)+s)duds.$$

Using formula (3.23) we can obtain the inequality

$$EH_n(\epsilon,t) \le 1 + O(\epsilon t) + \epsilon b_2 \int_0^t EH_n(\epsilon,s)ds$$

for some $b_2 > 0$. This implies (3.31).

4 Some Applications

Here we present asyptotic formulas for solutions of stochastically perturbed wave equations for vibration of a string and vibration of a circular membrane.

a) Vibration of a string. Non-stochastically perturbed equation for free vibration of a string is given by

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2}\ , \tag{4.1}$$

where the solution $u(t,x)$ is subject to:

$$u(t,0) = u(t,1) = 0, \text{ boundary conditions}, \tag{4.2}$$

$$u(0,x) = f(x),\ \frac{\partial u(0,x)}{\partial t} = g(x), \text{ initial conditions}. \tag{4.3}$$

The Fourier method allows us to write the solution in the form

$$u(t,x) = \sum_{k=1}^{\infty} r_k \sin(k\pi x)\cos(ak\pi t + \varphi_k), \tag{4.4}$$

where r_k and φ_k are determined by the initial conditions:

$$\left.\begin{aligned} r_k \cos\varphi_k &= 2\int_0^1 \sin(k\pi x) f(x)dx, \\ (a\pi k)\ r_k \sin\varphi_k &= -2\int_0^1 \sin(k\pi x) g(x)dx \end{aligned}\right\} \tag{4.5}$$

r_k's and φ_k's are called the amplitudes and the phases of the eigenoscillations of the string respectively, $k = 1, 2, \ldots$

We now consider the stochastic version of the wave equation, namely

$$\frac{\partial_\epsilon^2(t,x)}{\partial t^2} = a^2(y(\tfrac{t}{\epsilon}))\frac{\partial^2 u_\epsilon(t,x)}{\partial t^2} \tag{4.6}$$

with the same boundary and initial conditions (4.2) and (4.3). The Markov process $y(t)$ is the same as before. Let r_k's satisfy the condition

$$\sum_{k=1}^{\infty} r_k e^{(\pi k)^2 t} < \infty \text{ for all } \quad t > 0. \tag{4.7}$$

Then the distributions of the random field $u_\epsilon(\frac{t}{\epsilon}, x)$ coincides asymptotically with the distributions of the random field

$$\hat{u}_\epsilon(t,x) = \tag{4.8}$$

$$\sum_{k=1}^{\infty} r_k e^{k\pi V_k(t)} \sin(k\pi x)\cos[a_k(\tfrac{t}{\epsilon} + Z_0(t) + k\pi Z_k(t)) + \varphi_k]$$

where $a_k = \bar{a}k\pi$; $V_1(t), V_2(t), \ldots, Z_0(t), Z_1(t), \ldots$ are independent Wiener processes with mean zero and $EZ_0(t) = 2ct$ and $EV_k^2(t) = EZ_k^2(t) = ct$, $k = 1, 2, \ldots$, where c is as in Theorem 2.2.

b) **Vibration of a circular membrane**. The unperturbed equation in this case is of the form

$$\begin{gathered} \frac{\partial^2 u(t,x,y)}{\partial t^2} = a^2\Delta u(t,x,y), (x,y) \in G, \\ G = \{(x,y) : x^2 + y^2 < 1\}, \Gamma = \{(x,y) : x^2 + y^2 = 1\}. \end{gathered} \tag{4.9}$$

The boundary condition is:

$$u(t,x,y) = 0,\ (x,y) \in \Gamma, \tag{4.10}$$

and the initial conditions are

$$u(0,x,y) = f(x,y), \qquad \frac{\partial}{\partial t}u(0,x,y) = g(x,y). \tag{4.11}$$

The solution of (4.9) subject to (4.10) and (4.11) may be represented in the form

$$u(t,r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\{r_{n,m}\psi_{n,m}(r,\theta)\cos(a\lambda_{n,m}t+\varphi_{n,m}) \\ +r^*_{n,m}\psi^*_{n,m}(r,\theta)\cos(a\lambda_{n,m}t+\varphi^*_{n,m})\}, \tag{4.12}$$

here $\psi_{n,m}(r,\theta)=J_n(\lambda_{n,m}r)\cos n\theta$, $\psi^*_{n,m}(r,\theta)=J_n(\lambda_{n,m}r)\sin n\theta$, J_n is the Bessel function of the first kind of order n, $\lambda_{n,m}$ is the mth positive zero of the function J_n; $r_{n,m}$, $r^*_{n,m}$, $\varphi_{n,m}$, $\varphi^*_{n,m}$ are determined by the initial conditions; r,θ are the polar coordinates of the point (x,y).

Let $u_\epsilon(t,x,y)$ be the solution of the stochastically perturbed equation (4.9) in which a^2 is replaced by $a^2(y(\frac{t}{\epsilon}))$ with the same boundary and initials conditions (4.10) and (4.11). Then under the condition

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} r_{n,m}e^{\lambda^2_{n,m}t}<\infty \text{ for all } t>0,$$

the distributions of the random field $u_\epsilon(\frac{t}{\epsilon},x,y)$ coinsides asymptotically with the distributions of the random field

$$\begin{aligned}\widehat{u}_\epsilon(t,x,y)=&\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\{r_{n,m}\psi_{n,m}(r,\theta)\exp(\lambda_{n,m}V_{n,m}(t))\\ &\times\cos[\overline{a}\lambda_{n,m}(\tfrac{t}{\epsilon}+Z_0(t)+\lambda_{n,m}Z_{n,m}(t))+\varphi_{n,m}]\\ &+r^*_{n,m}\psi^*_{n,m}(r,\theta)\exp(\lambda_{n,m}V_{n,m}(t))\\ &\times\cos[\overline{a}\lambda_{n,m}(\tfrac{t}{\epsilon}+Z_0(t)+\lambda_{n,m}Z^*_{n,m}(t))+\varphi^*_{n,m}]\},\end{aligned} \tag{4.13}$$

where $r_{n,m}$, $r^*_{n,m}$, $\varphi_{n,m}$, $\varphi^*_{n,m}$ are the same as in formula (4.12), and $Z_0(t)$, $V_{n,m}(t)$, $Z_{n,m}(t)$, $Z^*_{n,m}(t)$ for $n\ge 0$ and $m\ge 1$ are independent Wiener processes with mean zero and $EZ_0^2(t)=2ct$ and $EV^2_{n,m}(t)=EZ^2_{n,m}(t)=E\{Z^*_{n,m}(t)\}^2=ct$, $n\ge 0$, $m\ge 1$.

Remark 4.1. Note that

$$E(r_ne^{n\pi V_n(t)})=r_ne^{n^2\pi^2\epsilon\ t/2}\ .$$

Suppose that there exists a number T for which

$$\sup_n\ r_ne^{n^2t}<\infty \text{ for } \quad t<T \text{ and } \quad \sup_n\ r_ne^{n^2t}=+\infty \text{ for } \quad t>T.$$

Then the series (4.8) converges for $t<\frac{2T}{\pi^2\epsilon}$ and diverges for $t>\frac{2T}{n^2\epsilon}$.
Let us consider the eigenoscillation of the form

$$u^{(n)}(t,x)=\exp\{n\pi V_n(\epsilon t)\}\cos\{an\pi t+Z_0(\epsilon t)+n\pi Z_n(\epsilon t)\}\sin(n\pi x).$$

Then its amplitude

$$\exp\{n\pi V_n(\epsilon t)\}$$

is unbounded in probability as $n \to \infty$. Thus random perturbations imply unbounded growth in the amplitudes of the eigenoscillations with high frequences. The condition

$$\sup_n \; r_n e^{n^2 t} < \infty$$

prevents the break down of the string in presence of random perturbutions.

Similar conclusions can be reached regarding the vibration of a membrane.

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