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Implementation of Control Policies for ARMAX Models

Nelson C. E. Townsend

University of Kansas
Department of Mathematics
Lawrence, Kansas

Abstract

This paper is a discussion of the characteristics of a computer program designed to implement the extended least squares algorithm for ARMAX models. The program simulates single input and single output ARMAX systems with arbitrary dependence on past states, controls, and noise. The controls implemented by the program are constant control, constant feedback control, white noise, excited control, diminishing excited control, optimal minimum variance control, and adaptive minimum variance control. The program is shown to be a valuable tool for examining parameter convergence, rate of convergence, and characteristics of the model, including stability and the strictly positive real condition.

1 Introduction

In the theory of parameter estimation, many algorithms have been advanced for the purpose of showing convergence of the estimates of parameters. Often these algorithms are defined in a way that glosses over the actual step by step procedure that the algorithm requires. A set of approximators are defined recursively, a trajectory is either given or evolves as the system advances, and a set of characteristics are defined which determine convergence and rate of convergence. However, the initial steps of the algorithm, involving questions of what to do with the initial values for the estimators, where to start the procedure , and what initial values the trajectory should have, are left out. Through the use of simulation, we are able to verify the results of commonly stated theorems which provide sufficient conditions for convergence of the extended least squares algorithm, while simultaneously examining the necessity and effect of these conditions.

A step by step procedure is developed and results of the procedure are discussed. Also, questions of Strictly Positive Real conditions are investigated and discussed. Our primary interest is the implementation of the extended least squares estimator algorithm for the auto-regressive, moving average, exogenous control or ARMAX model. Furthermore, we investigate the convergence and rate of convergence of the estimators of the parameters of these systems under various control policies. The control policies we focus our attention on are constant control, constant feedback control, white noise, excited control, diminishing excited control, optimal minimum variance control, and adaptive minimum variance control.

2 The Computer Algorithm

The computer program was designed to implement the extended least squares algorithm for ARMAX models. The extended least squares algorithm, sometimes referred to as the ELS algorithm, is also known as pseudolinear regression (PLR) or approximate maximum likelihood (AML). An ARMAX model is a linear relation with dependence on the past noise (w_i), states (y_i), and controls (u_i). Thus, the models we consider are of the form:

$$\begin{aligned} y_n &= -A_1 - \cdots - A_p y_{n-p} + \cdots \\ &\quad + B_1 u_{n-d} + B_2 u_{n-d-1} + \cdots + B_q u_{n-d-q+1} + \cdots \\ &\quad + C_1 w_{n-1} + \cdots + C_r w_{n-r} + w_n,\ n \geq 0; \end{aligned} \tag{2.1}$$

$$y_n = w_n = 0,\ n < 0;\ p \geq 0,\ r \geq 0,\ d \geq 0.$$

Here, y_i, u_i, and w_i are k x 1 real-valued vectors; A_i, B_i, and C_i are k x k real-valued matrices; p is the number of past inputs the system depends on; q is the number of past controls; r is the number of past noise inputs; and d is a delay. The particular computer program used to investigate the properties of the above system is restricted to the case of single input and single output. Furthermore, the type of control is restricted to the seven kinds mentioned previously. The noise inputs used in the simulation are randomly generated numbers from the normal distribution with mean 0 and variance 1.

The computer program employs an on-line approach to estimate the parameters. That is, as the trajectory evolves according to the model, the estimates evolve. Also, the estimate at each increment is saved for later evaluation. This aids greatly in comparing the rates of convergence of different control policies.

Before going further with the explanation of the computer program, it will be advantageous to introduce some definitions which will clarify the later discussion. Let

$$
\begin{aligned}
\theta &= [-A_1 \cdots - A_p B_1 \cdots B_q C_1 \cdots C_r]^T; \\
\phi_n &= [y_n \cdots y_{n+p-1} u_{n-d} \cdots u_{n-q-d+1} w_n \cdots w_{n-r+1}]^T; \\
\widehat{w}_n &= y_n - \theta_n \phi_{n-1}, \text{ where } \widehat{w} \text{ is called an } \textit{a priori} \text{ estimate of noise, i. e. the algorithm estimates the noise of the present state using the past states of the system;} \\
\widehat{\phi}_n &= [y_n \cdots y_{n+p-1} u_{n-d} \cdots u_{n-q-d+1} \widehat{w}_n \cdots \widehat{w}_{n-r+1}]^T; \text{ and} \\
\widehat{\theta}_{n+1} &= \widehat{\theta}_n + a_n P_n \phi_n (y_{n+1}^T - \phi_n^T \widehat{\theta}_n), \text{ where} \\
a_n &= (1 + \phi_n^T P_n \phi_n)^{-1} \text{ and} \\
P_{n+1} &= (P_n - a_n P_n \phi_n \phi_n^T P_n).
\end{aligned}
\tag{2.2}
$$

Note that P_{n+1} is derived using the Matrix Inversion Lemma. [See: [CG], pg. 90, for the derivation of P_n and a_n.]

Throughout our investigation, we assume that the state of the system and the controls of the system are observable. However, we assume that all coefficients and the noise terms are unobservable. Note that we can rewrite the previous ARMAX model in the linear regression form

$$y_{n+1} = \theta^T \phi_n + w_{n+1},$$

and our purpose is to estimate θ by $\widehat{\theta}_n$.

Now, we return to the discussion of the computer algorithm. First, a state is generated. Since we take $y_0 = 0$ and $w_0 = 0$, we have

$$y_1 = B_1 u_1 + w_1.$$

Future values may involve more terms as the trajectory, controls, and noise take non-zero values. An *a priori* estimate of the noise is made. Next, the first estimate of the parameters of the system is made. Since $\widehat{\theta}_n$ is defined recursively, an initial value for $\widehat{\theta}_1$ is required. Since no *a priori* information about θ is assumed, we arbitrarily choose

$$\widehat{\theta}_1 = [0 \cdots 0].$$

Another parameter that is defined recursively is P_n. Note that P_n is defined non-recursively by

$$P_n = \sum_{k=1}^{n} \phi_k \phi_k^T.$$

Unfortunately, since

$$
\begin{aligned}
\phi_1 &= [y_1\, 0 \cdots 0\, u_1\, 0 \cdots 0\, w_1\, 0 \cdots 0]^T \text{ and} \\
P_1 &= \left[\phi_1 \phi_1^T\right]^{-1},
\end{aligned}
$$

it is possible that $\phi_1\phi_1^T$ will not be invertible. Then, P_1 would not be defined. Then, since P_n is defined recursively, P_n would not be defined. To avoid this problem, we choose

$$P_0 = \alpha I_{p+q+r}\,, \text{ where } 0 < \alpha < 1/e.$$

Then, P_0 is clearly invertible, and the recursive definition of P_n is well-defined. This modification does not affect the asymptotic behavior of the system. [See: [CG], pg. 91]

After $\widehat{\theta}_1$ has been defined and P_0 has been calculated, the recursion becomes straightforward. The next state of the system is generated. The noise is estimated, etc.

3 Output of the Program

The program displays the system trajectory, the system's control, the estimates of the parameters (in solid lines), the true parameters (in dotted lines), and the cost function of the path, control, and their total. The cost function depends on the number of steps taken in the simulation (k) and the trajectory of the path or control.

The cost function employed for the path is

$$\mathcal{C}_{\text{path}} = \tfrac{1}{k}\sum_{i=1}^{k} y_i^2.$$

That is, the cost of the path is its average deviation from zero sqared. This is the cost that is minimized under the optimal minimum variance control policy. The cost is computed at every stage of the evolving system. The cost of the control is similar:

$$\mathcal{C}_{\text{control}} = \tfrac{1}{k}\sum_{i=1}^{k} u_i^2.$$

The total cost is simply the sum of the costs for the path and control:

$$\mathcal{C}_{\text{total}} = \mathcal{C}_{\text{path}} + \mathcal{C}_{\text{control}}.$$

As was noted earlier, the program maintains cumulatvie information of the costs, state, and control.

4 Main Results

The following discussion is based on the results of numerous simulations conducted using the previously discussed computer program.

4.1 Constant Control

Convergence of the parameters is good under the constant control policy when the system is of the form

$$\begin{aligned} y_n = & -A_1 - \cdots - A_p y_{n-p} + \cdots \\ & +B_1 u_{n-d-1} + \cdots \\ & +C_1 w_{n-1} + \ldots + C_r w_{n-r} + w_n, \end{aligned} \tag{4.1}$$

i.e., there is only one parameter to estimate for the controls. In the case that the system depends on more than one past control the algorithm is able to identify the sum of parameters for past controls. Suppose the system has the form

$$\begin{aligned} y_n = & -A_1 - \cdots - A_p y_{n-p} + \cdots \\ & +B_1 u_{n-d-1} + \cdots + B_q u_{n-d-q} + \cdots \\ & +C_1 w_{n-1} + \cdots + C_r w_{n-r} + w_n, \text{ where } q > 1. \end{aligned} \tag{4.2}$$

Since every control (u_i) is identical, the system can be re-written in the form

$$\begin{aligned} y_n = & -A_1 - \cdots - A_p y_{n-p} + \cdots \\ & +u_{n-d-1}(B_1 + \cdots + B_q) + \cdots \\ & +C_1 w_{n-1} + \cdots + C_r w_{n-r} + w_n, \text{ where } q > 1. \end{aligned} \tag{4.3}$$

Thus, the extended least squares algorithm is able to identify $\sum_{i=1}^{q} B_i$, but not B_1, B_2, …,B_q individually.

4.2 Constant Feedback

A problem in identification similar to (4.1) arises in the constant feedback policy. The control is generated using the most recent state of the system. Thus, we have

$$u_i = cy_i, \text{ for some constant } c.$$

Then, a system depending on one past state and one control,

$$y_n = A_1 y_{n-1} + B_1 u_{n-1} + w_n,$$

can be rewritten as

$$y_n = (A_1 + B_1 c) y_{n-1} + w_n.$$

Hence, the algorithm is only able to identify the quantity $(A_1 + B_1 c)$. Similar results occur for models of arbitrary size.

4.3 White Noise

If the control is of this type then the estimates of the parameters converge if the system is stable. Also, for this control policy, the rate of convergence is high. Under this control policy, the question of the necessity of the so called Strictly Positive Real condition, or SPR, can be investigated. Many systems were created with the SPR condition "just barely" violated. In these cases, the estimates of the parameters did not converge in under 20,000 time steps. Although, this does not settle the question of necessity, the program does allow the user to gain some familiarity with systems in which the SPR condition does not hold.

4.4 Excited Control

This control, similar to the white noise control, allows for the parameters to converge when the system is stable. This control is especially useful if the control coefficients are relatively small. In this case, as the estimates for the control parameters get closer to their "true" values, the values for the control increase. Thus, estimates and rates of convergence for the control coefficients are improved. However, the other parameter estimates converge more slowly under this policy than the white noise policy, for instance. Also, the cost for controlling the system is quite high when the control coefficients are close to zero.

4.5 Diminishing Excited Control

Under this policy the estimates of the control coefficients, will begin to converge. In some cases, as the control diminishes, the estimates of the control parameters grow worse and worse and do not converge. However, in many cases, the other estimates of the parameters do converge. As expected, the cost for control is relatively small. However, the total cost may be high because the system is not being controlled optimally.

4.6 Optimal Minimum Variance

This control policy exhibits behavior similar to constant feedback control. That is because this control is similar to the constant feedback control policy. We assume that the parameters of the system are known in order to compute the control. Thus, estimating the parameters with the ELS algorithm is of marginal interest. The equation for the control is

$$u_n = \frac{1}{B_1}\left[\sum_{i=1}^{p}(A_i + C_i)y_{n-i} + \sum_{i=2}^{q} B_i u_{n-i}\right].$$

Not surprisingly, certain functions of the true parameters and their final estimates are nearly equal. For instance, in many cases we have

$$\frac{A_1+C_1}{B_1} \approx \frac{\widehat{A}_1+\widehat{C}_1}{\widehat{B}_1}.$$

4.7 Adaptive Minimum Variance Control

This policy is self tuning, but not consistent. However, certain functions of the true parameters and their estimates converge to similar values. For instance, in many models we have

$$\frac{A_1+C_1}{B_1} \approx \frac{\widehat{A}_1+\widehat{C}_1}{\widehat{B}_1}.$$

An interpretation of the reason for the estimates to self-tune and not converge is that as the estimates converge to the correct values, the values used for the control do not control the system optimally. This causes the estimates to be incorrect. Thus, the estimates self-tune and do not converge. Under the adaptive minimum variance control policy, we do not assume that the parameters of the system are known. Instead, the estimates of the system's parameters are used to compute the control. Thus, the equation for the control is similar to the optimal minimum variance case and has the form

$$u_n = \frac{1}{\widehat{B}_1}\left[\sum_{i=1}^{p}(\widehat{A}_i + \widehat{C_i}y_{n-i} + \sum_{i=2}^{q}\widehat{B_i}u_{n-i}\right].$$

5 Further Investigation

Investigating the behavior of the white noise control policy under conditions when the SPR condition are nearly violated or just barely violated is one avenue of further investigation.

The self-tuning aspects of the adaptive minimum variance control are very interesting. A possible avenue of investigation is the distribution of the limit points of the parameter estimates under this control policy.

Further aspects of the excited control are also open to investigation.

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