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Identification of Parameters with Convergence Rate for Bilinear Stochastic Differential Equations

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Abstract

We consider the parameter identification problem for both drift and diffusion coefficients of bilinear stochastic differential equations. A strongly consistent estimator for the coefficient of the drift is known for the case in which we have continuous observation of the state. The values of the parameters for the diffusion coefficients are known after an arbitrarily small positive interval of time: formulas for the actual computation of their values are given. For the case in which we observe the state only at discrete moments of time, we discretize the formulas. It is shown that the discretization of these estimators does not converge to the parameters but to quantities that depend both on the values of the parameters and on the disretization step. The expression of these quantities are given explicitly. Results on the rate of convergence of these estimators are given. Finally we give strongly consistent estimators for the discrete observations case.

1 Introduction

We consider the bilinear stochastic differential equations, for $i = 1, .., d$,

$$dX_i(t) = b_i\, X_i(t)dt + \textstyle\sum_{j=1}^d \sigma_{ij}\, X_i(t)\, dW_j(t), \qquad X_0 = x_0, \tag{1.1}$$

where $b = (b_1, .., b_d)'$ is a constant vector, $\sigma = (\sigma_{ij})_{i,j=1}^d$ a constant matrix, and W a d-dimensional Brownian motion. This kind of equations has found applications in financial mathematics as models for the evolution of prices in the stock market. In particular, an application to problems of optimal investment and consumption (see [K]) shows optimal policies that depend explicitly on the values of b and $A := \sigma\sigma'$. These parameters (average rate of return of the stocks and volatility of the market) are not known by the investor, who has the need for estimating their values in such a way that he/she can define the policy of investment and consumption. In this paper strongly consistent estimators for b and A are given both for the case in which we observe $X(t) = (X_1(t), .., X_d(t))'$ for every $t \geq 0$ and for the case in which the observations are made only at discrete moments $t \in \{t_k^\delta := k\delta\}_{k=0}^{\infty}$, where δ is a positive constant. For the case in which we have continuous observations a strongly consistent estimator of b is known and so is its rate of convergence (see [DPD1], [DPD2]). We give a discretized version of this estimator and show that the discretization introduces a bias that depends on the discretization step and that can be found explicitly. This fact allows us to define a strongly consistent estimator for the discrete observation case. The value of A is known after an arbitrarily small positive period of time for the case in which we have continuous observations; a formula for the computation is given. Just like in the previous case the discretization of this formula gives us a biased estimator for A; the value of the biased is found and a strongly consistent estimator for the discrete observation case is given. Finally results on the rate of convergence of the discretized estimators are given. The paper is organized as follows: in Section 2 estimators for both b and A are given for the case with continuous observations. The convergence of their discretization with results on the rate of convergence is given in Section 3. The results of the previous section are then used in Section 4 to prove the strong consistency of the estimators that are given for both b and A for the case in which we have discrete observations. The results are then tested through simulation in Section 5.

2 Continuous observations

Let us consider the case in which we want to estimate the parameters b and A in (1.1) and we have continuous observations of the state $X(t)$. The estimator of b, given for the case $d = 1$ in [BF], [DPD1], [K] and [S], can be used also for the case $d > 1$. In fact, we have the following result

Proposition 2.1 *For every i=1,..,d, let $\bar{b}_i(t)$ be defined for $t > 0$ by*

$$\bar{b}_i(t) = \frac{1}{t}\int_0^t \frac{dX_i(s)}{X_i(s)} \tag{2.1}$$

We have that these estimators are strongly consistent, i.e.

$$\lim_{t\to\infty} \bar{b}_i = b_i \qquad a.s. \tag{2.2}$$

Proof. Using (1.1) we get

$$\bar{b}_i(t) = \frac{1}{t}\int_0^t \frac{dX_i(s)}{X_i(s)} \tag{2.3}$$

$$= \frac{1}{t}\int_0^t b_i\, ds + \sum_{j=1}^d \sigma_{ij}\, dW_j(s) = b_i + \sum_{j=1}^d \sigma_{ij}\, \frac{W_j(t)}{t}$$

and the result follows from the strong law of large numbers for Brownian motion.

□

Remark 2.1 *The rate of convergence of the estimator is discussed in [DPD2] using the law of iterated logarithm*

We can now move our attention to A: we observe that there is no need to estimate the entries of A. In fact these are known adter an arbitrarily small period of time of positive length provided that we have continuous observations of the state. It is enough to evaluate a stochastic integral to get the exact values

Proposition 2.2 *For every i,j = 1,..,d and* $t > 0$

$$A_{ij} = \frac{1}{t}\int_0^t \frac{d(X_i(s)X_j(s))}{X_i(s)X_j(s)} - \bar{b}_i(t) - \bar{b}_j(t) \tag{2.4}$$

Proof. Applying Ito's differential rule we get

$$d(X_i(s)X_j(s)) = X_i(s)\,dX_j(s) + X_j(s)\,dX_i(s) \tag{2.5}$$

$$+\frac{1}{2}tr(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X_i(s)\,\sigma_{i1} & ...X_i(s)\,\sigma_{id} \\ X_j(s)\,\sigma_{j1} & ...X_j(s)\,\sigma_{jd} \end{bmatrix}\begin{bmatrix} X_i(s)\,\sigma_{i1} & ...X_i(s)\,\sigma_{id} \\ X_j(s)\,\sigma_{j1} & ...X_j(s)\,\sigma_{jd} \end{bmatrix}')\,ds$$

$$= X_i(s)\,dX_j(s) + X_j(s)\,dX_i(s) + \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk}X_i(s)X_j(s)\,ds$$

dividing both sides by $X_i(s)X_j(s)$ and integrating from 0 to t we get the result.

□

3 Discrete observations I

In this section we define estimators for b and A for the case in which we have observations only at discrete moments $\{t_k^\delta := k\delta\}_{k=0}^{\infty}$, $\delta > 0$. We exploit the fact that explicit solutions for equations (1.1) are known for $i = 1,..,d$, and given by ([KS])

$$X_i(t) = x_0\,\exp\{(b_i - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2)t + \sum_{j=1}^{d}\sigma_{ij}W_j(t)\} \tag{3.1}$$

We proceed by giving a discretization of the estimators in (2.1)and (2.4); let

$$\hat{b}_i^\delta(k) := \frac{1}{k\delta}\sum_{l=1}^{k}\frac{X_i(t_l^\delta) - X_i(t_{l-1}^\delta)}{X_i(t_{l-1}^\delta)} \tag{3.2}$$

$$\hat{A}_{ij}^\delta(k) := \frac{1}{k\delta}\{\sum_{l=1}^{k}\frac{X_i(t_l^\delta)X_j(t_l^\delta) - X_i(t_{l-1}^\delta)X_j(t_{l-1}^\delta)}{X_i(t_{l-1}^\delta)X_j(t_{l-1}^\delta)} \tag{3.3}$$

$$-\frac{X_i(t_l^\delta) - X_i(t_{l-1}^\delta)}{X_i(t_{l-1}^\delta)} - \frac{X_j(t_l^\delta) - X_j(t_{l-1}^\delta)}{X_j(t_{l-1}^\delta)}\}$$

Remark 3.1 *If we define* $\Phi_i^\delta(k)$ *by*

$$\Phi_i^\delta(k) := \frac{X_i(t_k^\delta) - X_i(t_{k-1}^\delta)}{X_i(t_{k-1}^\delta)} \tag{3.4}$$

then the definitions given in (3.2) and (3.3) are equivalent to the following

$$\hat{b}_i^\delta(k) := \frac{1}{k\delta}\sum_{l=1}^{k}\Phi_i^\delta(l) \tag{3.5}$$

$$\hat{A}_{ij}^\delta(k) := \frac{1}{k\delta}\sum_{l=1}^{k}\Phi_i^\delta(l)\Phi_j^\delta(l) \tag{3.6}$$

Remark 3.2 *By defintion of Ito stochastic integral for* $\delta \to 0$ *we have that* $\hat{b}_i^\delta(k)$ *and* $\hat{A}_{ij}^\delta(k)$ *converge to* $\bar{b}_i(t_k)$ *and* A_{ij} *respectively.*

The estimators defined in (3.5) and (3.6) converge almost surely, as k approaches infinity, to quantities that depend not only on b and A but also on δ.

Theorem 3.1 *For every* $\delta > 0$ *and i=1,..,d*

$$\lim_{k\to\infty} \hat{b}_i^\delta(k) = \frac{\exp\{\delta\, b_i\} - 1}{\delta} \qquad a.s. \tag{3.7}$$

and for every $\delta > 0$ *and i,j=1,..,d*

$$\lim_{k\to\infty} \hat{A}_{ij}^\delta(k) = \frac{\exp\{\delta(b_i + b_j) + A_{ij})\} - \exp\{\delta\, b_i\} - \exp\{\delta\, b_j\} + 1}{\delta} \quad a.s. \tag{3.8}$$

Proof. Let $Y_i(t) := \log(X_i(t))$; it follows from (3.1) that at time t_{k+1} the distribution of Y_i is

$$Y_i(t_{k+1}) \;\sim\; Y_i(t_k) + (b_i - \frac{1}{2}A_{ii})\delta + \sqrt{\delta}\sum_{m=1}^{d}\sigma_{im}w_m(k+1) \tag{3.9}$$

where $\{w(k+1) := (w_1(k+1), .., w_d(k+1))\}_{k=0}^\infty$ is a sequence of independent Gaussian random vectors and the symbol $\sim$ indicates that the random variables on the two sides have the same distribution. ¿From this it follows that $\Phi_i^\delta(l)$, defined in (3.4), is such that

$$\Phi_i^\delta(l) \;\sim\; \exp\{(b_i - \tfrac{1}{2}A_{ii})\delta + \sqrt{\delta}\textstyle\sum_{m=1}^{d}\sigma_{im}w_m(k+1)\} \,-\, 1 \tag{3.10}$$

If we fix i and δ, we have that the sequence $\{\frac{\Phi_i^\delta(l)}{\delta}\}_{l=1}^\infty$ is a sequence of independent, identically distributed random variables. Applying the strong law of large numbers (see [Sh]) and (3.5) we get

$$\lim_{k\to\infty} \hat{b}_i^\delta(k) = \lim_{k\to\infty} \frac{1}{k}\sum_{l=1}^{k} \frac{\Phi_i^\delta(l)}{\delta} = \frac{\exp\{\delta\, b_i\} - 1}{\delta} \tag{3.11}$$

that proves (3.7).
For proving (3.8) observe that

$$\Phi_i^\delta(l)\,\Phi_j^\delta(l) = \exp\{\delta\,(b_i + b_j - \tfrac{1}{2}(A_{ii} + A_{jj})) \tag{3.12}$$

$$+\sqrt{\delta}\sum_{m=1}^{d}(\sigma_{im} + \sigma_{jm})w_m(l)\} - \Phi_i^\delta(l) - \Phi_j^\delta(l) + 1$$

and similarly to the previous case using (2.10), (2.16) and the strong law of large numbers we have the result.

□

In order to get results on the rate of convergence of these estimators, we use a version of the law of iterated logarithm proved by Hartman and Wintner (see [Sh], pages 372-374)

Theorem 3.2 *Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent identically distributed random variables with $E\{\xi_i\} = 0$ and $E\{\xi_i^2\} = \gamma^2 > 0$; let $S_n = \sum_{i=1}^n \xi_i$. Then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2\gamma^2 n \log\log n}} = 1 \quad a.s. \tag{3.13}$$

Using the previous theorem, we can prove the following results

Proposition 3.3 *For every $i = 1,..,d$, δ and $\epsilon > 0$ we have that*

$$\limsup_{k\to\infty} \frac{|\hat{b}_i^\delta(k) - b_{i,\delta}^\infty|}{k^{\epsilon-\frac{1}{2}}} \le \sqrt{2}\gamma_{i,\delta}^b \quad a.s. \tag{3.14}$$

where

$$b_{i,\delta}^\infty = \frac{\exp\{\delta b_i\} - 1}{\delta} \tag{3.15}$$

and

$$\gamma_{i,\delta}^b = \frac{E\{(\Phi_i^\delta)^2\}}{\delta^2} - (b_{i,\delta}^\infty)^2 \tag{3.16}$$

Moreover, for every $i, j = 1,..,d$, $\delta > 0$ and ϵ

$$\limsup_{k\to\infty} \frac{|\hat{A}_{ij}^\delta(k) - A_{i,j,\delta}^\infty|}{k^{\epsilon-\frac{1}{2}}} \le \sqrt{2}\gamma_{i,j,\delta}^A \quad a.s. \tag{3.17}$$

where

$$A_{i,j,\delta}^\infty = \frac{\exp\{\delta((b_i + b_j) + A_{ij})\} - \exp\{\delta\, b_i\} - \exp\{\delta\, b_j\} + 1}{\delta} \tag{3.18}$$

and

$$\gamma_{i,j,\delta}^A = \frac{E\{(\Phi_i^\delta\,\Phi_j^\delta)^2\}}{\delta^2} - (A_{i,j,\delta}^\infty)^2 \tag{3.19}$$

Proof.

$$P\{\limsup_{k\to\infty} \frac{|\hat{b}_i^\delta(k) - b_{i,\delta}^\infty|}{k^{\epsilon-\frac{1}{2}}} \le \sqrt{2}\gamma_{i,\delta}^b\}$$

$$\ge P\{\limsup_{k\to\infty} \frac{|\hat{b}_i^\delta(k) - b_{i,\delta}^\infty|\sqrt{k}}{\sqrt{\log\log k}} \le \sqrt{2}\gamma_{i,\delta}^b\}$$

$$= P\{\limsup_{k\to\infty} \frac{|\frac{1}{k}\sum_{l=1}^k \frac{\Phi_i^\delta(l)}{\delta} - \frac{1}{k}\sum_{l=1}^k b_{i,\delta}^\infty|\sqrt{k}}{\sqrt{\log\log k}} \le \sqrt{2}\gamma_{i,\delta}^b\}$$

$$= P\{\limsup_{k\to\infty} \frac{|\sum_{l=1}^k (\frac{\Phi_i^\delta(l)}{\delta} - b_{i,\delta}^\infty)|}{\sqrt{2\,\gamma_{i,\delta}^b\,k\,\log\log k}} \le 1\} = 1 \quad a.s.$$

by theorem 3.2. The proof of the second part of the statement is analogous.

□

Remark 3.3 *The expected values in 3.16 and 3.19 can be easily evaluated using the fact that if* $X \sim N(a,\sigma^2)$ *then* $E\{\exp\{tX\}\} = \exp\{at + \frac{\sigma^2 t^2}{2}\}$.

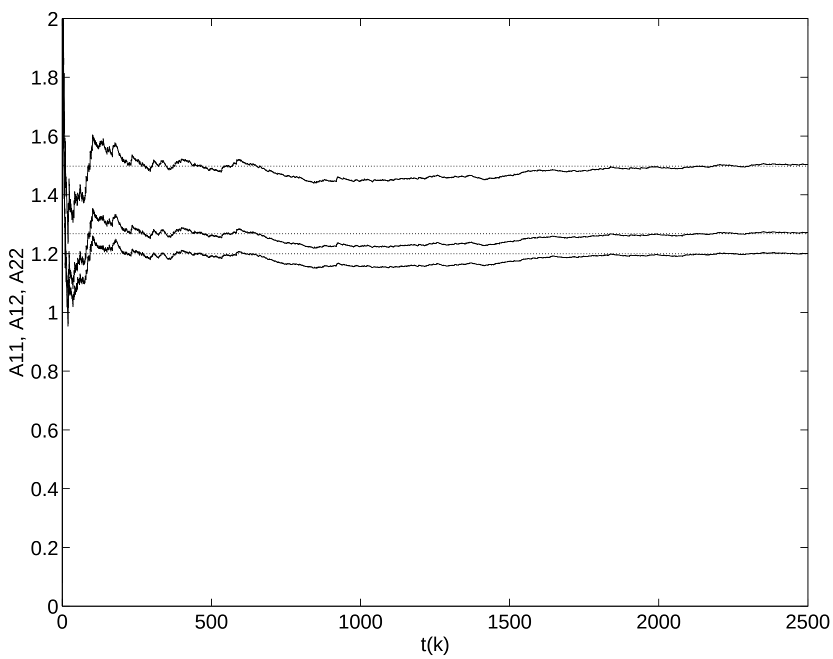

Fig. 1: delta=0.1

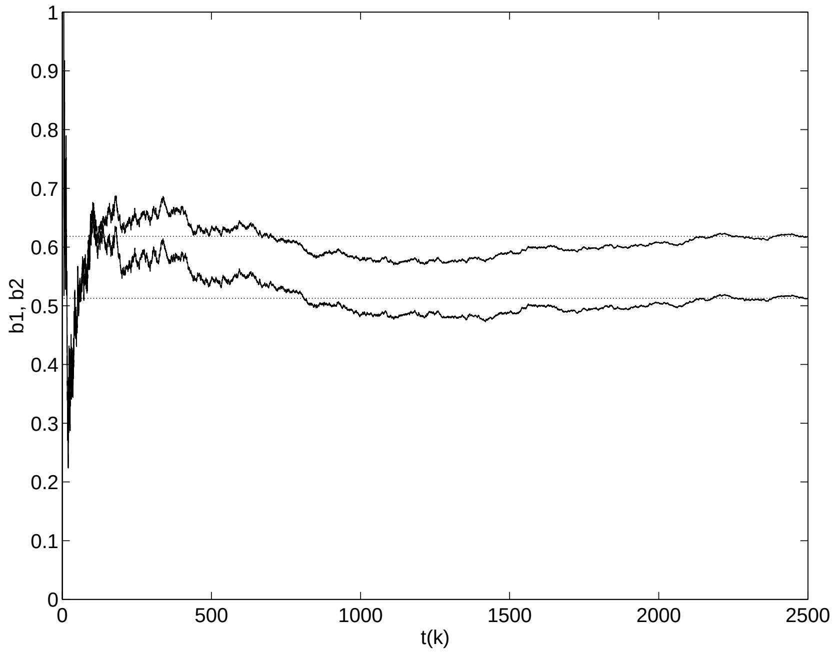

Fig. 2: delta=0.1

4 Discrete observations II

Let us define strongly consistent estimators for b and A in the case in which we have discrete obsrevations. For $i, j = 1, .., d$, let $\hat{\hat{b}}_i^\delta(0)$ and $\widehat{\hat{A}_{ij}^\delta}(0)$ be set equal to arbitrary constants. Let Q and R denote

$$Q(\delta, i, k) \; = \; \delta \, \hat{b}_i^\delta(k) \, + \, 1 \tag{4.1}$$

$$R(\delta, i, j, k) \; = \; \delta \, \hat{A}_{ij}^\delta(k) \, - \, 1 \, + \, \exp\{\delta \, \hat{b}_i^\delta(k)\} \, + \, \exp\{\delta \, \hat{b}_j^\delta(k)\} \tag{4.2}$$

and define recursively for every $i = 1, .., d$ and $k > 0$

$$\hat{\hat{b}}_i^\delta(k) := \begin{cases} \dfrac{\log\{Q(\delta, i, k)\}}{\delta} & if \; Q(\delta, i, k) > 0 \\ \hat{\hat{b}}_i^\delta(k-1) & otherwise \end{cases} \tag{4.3}$$

and for $i, j = 1, .., d$ and $k > 0$

$$\widehat{\hat{A}_{ij}^\delta}(k) := \begin{cases} \dfrac{\log\{R(\delta, i, j, k)\}}{\delta} - \hat{b}_i^\delta(k) - \hat{b}_j^\delta(k) & if \; R(\delta, i, j, k) > 0 \\ \widehat{\hat{A}_{ij}^\delta}(k-1) & otherwise \end{cases} \tag{4.4}$$

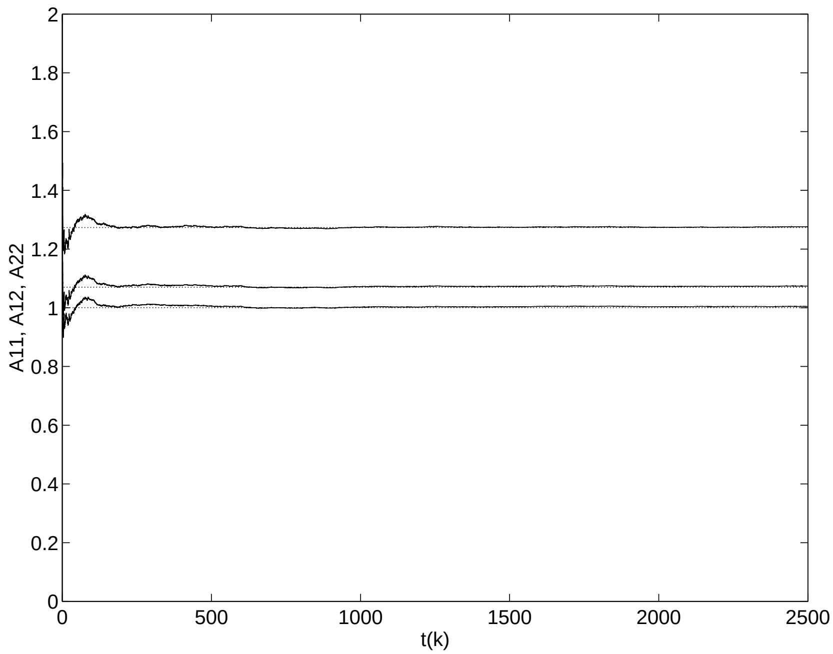

Fig. 3: delta=0.01

We have that

Theorem 4.1 *For every $\delta > 0$ and i=1,..,d*

$$\lim_{k\to\infty} \widehat{b_i^\delta}(k) = b_i \qquad a.s. \tag{4.5}$$

and for every $\delta > 0$ and i,j=1,..,d

$$\lim_{k\to\infty} \widehat{A_{ij}^\delta}(k) = A_{ij} \qquad a.s. \tag{4.6}$$

i.e., the estimators defined above are strongly consistent.

Proof. The strong consistency of the estimators follows straightforwardly from Theorem 2.3.

□

5 Simulation

In this section we analyze, through a simulation, the effect that different choices of δ (length of the interval between observations) have on the discretized estimators illustrating therefore the results of theorem 3.1. We generate a sample path of the solution of (1.1) using the property mentioned

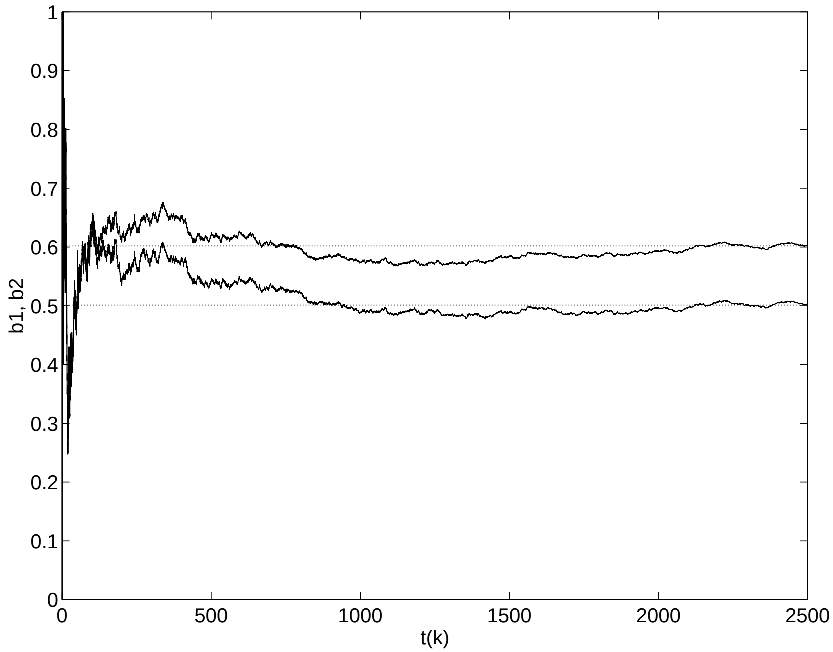

Fig. 4: delta=0.01

in 3.9. We simulate the system for the case in which $d = 2$; the discretization step is $\Delta t = 0.001$ and the time horizon is $T = 2500$. Using the sample paths for $X_1(t)$ and $X_2(t)$ obtained from this simulation, we compute the values of the estimators for the cases in which we have

1) 100 observations per unit of time (i.e. $\delta = 0.01$)
2) 10 observations per unit of time (i.e. $\delta = 0.1$)

The values of the parameters that have been used are

$$b := \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 0.7 & 0.7 \\ 1 & 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.98 & 1.05 \\ 1.05 & 1.25 \end{bmatrix}$$

The theoretical values, obtained in theorem 3.1, for the three cases are given in Table 1.

δ	$b_{1,\delta}^{\infty}$	$b_{2,\delta}^{\infty}$	$A_{1,1,\delta}^{\infty}$	$A_{1,2,\delta}^{\infty}$	$A_{2,2,\delta}^{\infty}$
.01	.6018	.5013	1.0003	1.0702	1.2730
.1	.6184	.5127	1.1991	1.2675	1.4978

Table 1

The figures 1-4 show that the results that are obtained are exactly what we are expecting from the theory.

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