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Alexandre Grothendieck's EGA V
Part VII:
Axiomatization of Some Geometric Results

(Interpretation and Rendition of his 'prenotes')

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§15 Axiomatization of Some Geometric Results

I think overall, of those results from No. 2 to 8 which are all mostly true under more general conditions than for the family of hyperplanes (or for hypersurfaces of given degree) in projective space. It seems to me the right time to adopt such an axiomatic point of view as soon as we re-edit the first sections. I am not quite sure right now if we can make such a generalization in this sense of Bertini-Zariski (therefore of the results of No. 4 and 6) and I wrote about it to the authorities (Serre, Zariski) to inquire if they knew of such an extension; I have anyway the impression that the hypotheses of simple differential nature such as given below should suffice to imply Bertini-Zariski. If these authorities could not inform us in a satisfactory manner, we should try to clear the matter up by ourselves.

We start with a commutative diagram (D) of morphisms of finite presentation

$$\begin{array}{ccc} P & \longleftarrow & \mathcal{P} \\ \downarrow & & \downarrow \\ S & \longleftarrow & G \end{array} \tag{D}$$

(in the case of the principal mapping, P is a projective fibration, G a deduced grassmanian and $\mathcal{P}$ the incidence prescheme. In most important cases the corresponding morphism $\mathcal{P} \to P \times_S G$ will be a closed immersion and we shall consider G as a parameter scheme of a family of closed fiber subpreschemes of P over S, more precisely if $\xi \in G$ then $\mathcal{P}_\xi$ is a closed subprescheme of $P_{sk(\xi)}$ where s is the point of S under ξ. (Besides for most statements in this context we have certainly $S = Spec(k)$). In the general case we may again consider G as a parameter scheme of a family of preschemes over the fibers of P over S with $\mathcal{P}_\xi$ over $P_{sk(\xi)}$ corresponding to ξ . Of course, in place of taking for ξ an (absolute) point of G we may also take a point with values in a S-prescheme T, and we obtain then $\mathcal{P}_\xi \to B_T$ (T-morphism which is a closed immersion in the case at first considered).

If $f\colon X \to P$ is a morphism, we set $\mathcal{X} = X \times_P \mathcal{P}$ and we obtain a diagram of the same type as the previous square.

$$\begin{array}{ccc} X & \longleftarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \longleftarrow & G \end{array}$$

It is therefore evident that all the questions studied in No. 2 to 8 preserve their meaning in the general context that we just stated and it is time to draw the axiomatic conditions that will ensure the conclusions drawn in the above Nos.

We shall assume that *P and G are flat over S*, G to be *with geometrically irreducible fibers* (to be able to consider the generic points!) *of dimension N, the morphism $\mathcal{P} \to P$ is assumed to be smooth with geometrically irreducible fibers of dimension $N - m$*. Therefore the morphism $\mathcal{X} \to X$ has the same properties. All the properties mentioned here and later are stable under base change over S and can in particular be applied to the fibers.

Let us first assume $S = Spec(k)$. Let Z be a closed subset of X of dim d, so its inverse image $\mathcal{Z}$ in $\mathcal{X}$ is a closed subset of dimension $d + (N - m) = N + d - m$. If $d < m$ then $\mathcal{Z}$ is of dimension $< N$ so that $\mathcal{Z} \to G$ cannot be dominant, therefore if η is the generic point of G we have $\mathcal{Z}_\eta = \phi$; indeed this argument shows even (by replacing Z by $\overline{f(Z)}$) that if dim $f(Z) < m$ then $\mathcal{Z}_\eta = \phi$. We want a condition on (D) ensuring that if dim $f(Z) \geq m$, then $\mathcal{Z}_\eta \neq \phi$. It seems that the condition $\mathcal{Z}_\eta \neq \phi$ must create a primitive axiom of the situation (in the No. 2.2 it resulted from a global argument rather special): *for every closed irreducible subset Z of P of dimension m, $\mathcal{Z}_\eta \neq \phi$.*

Let us take again a closed subset Z of X such that dim $f(Z) \geq m$, we see that $\mathcal{Z}_\eta \to G$ is dominant and consequently $\mathcal{Z}_\eta$ is of dimension equal to dim $\mathcal{Z}$ − dim G = dim $Z - m$.

These properties allow us to develop in the present context the results corresponding to those of Nos. 2.1 to 2.11. There is a condition on (D) ensuring the validity of 2.12, i.e. that if X is smooth then so is $\mathcal{X}$, if we assume $f\colon X \to P$ unramified. We shall assume now that P is *smooth over* k, $\mathcal{P} \to P \times_k G$ quasi-finite, and that the following condition is satisfied (where we assume k algebraically closed): for every $x \in P(k)$ and for every vector subspace V of the tangent space $T_x(P)$ to P at x, of dimension $n \geq m$, we consider the set $E(x, V)$ of all $\xi \in G(k)$ such that $\mathcal{P}_\xi$ has a point over x that does *not* satisfy the following set of conditions:

$\mathcal{P}_\xi$ is smooth at z, the tangent mapping to $\mathcal{P}_\xi \to P$ at z: $T_z(\mathcal{P}) \to T_x(P)$ is injective (i.e. $\mathcal{P}_\xi \to P$ is unramified at z) and its image is "transversal" to V, i.e. its sum with V is $T_x(P)$.

Then $E(x, V)$ (which we know to be the trace of a well defined constructible set of G on $G(k)$) *is of dimension* $\leq N - n - 1$. Subsequently to this condition, the application of the Jacobian criterion and a dimension count imply that the closed subset E of points x of $\mathcal{X}$ such that $\mathcal{X} \to G$ is non-smooth at x **or** $\mathcal{P} \to G$ is not smooth at $f(x)$ **or** $\mathcal{P} \to P$ is ramified at $f(x)$ and of dimension $\leq n + (N - n - 1) = N - 1$, ($X$ being smooth everywhere of dimension n). Therefore dim $E < N$ = dim G, so that $E_\eta = \phi$ and á fortiori X_η is smooth over $k(\eta)$. These last facts being established, we may right away deduce the validity of the obvious variants of 2.12 to 2.18 in the present context.

The passage in No. 4 from a generic section to a general section and the developments of No. 5 are obviously valid in the present context (but are at this point tautologies or repetitions of paragraphs 8, 9, 12 that we hesitate to formulate them).

It is also the same for the developments of 7.1, valid anyway if k is algebraically closed (and even if k is simply infinite and if G is rational over k) and also for the special cases 7.2, 7.3; as for the results in 7.4 they are clearly an application of special nature related to hyperplane sections. As I said before, the Nos. 3 and 6 are pending to the extension of the theorem of Zariski.

It would remain to extend also the results of No. 8 (reconsidered in No. 12) which then could take on such a more pleasant aspect. I even suggest to you to begin with the formulation of these results in this context and to try to go as far as possible in this direction. I have the impression that it should be possible to recover at least what is not a direct consequence of 8.7. c) (even we could yet attempt to generalize the axiomatic conditions that should imply a variant of 8.7.c). I restrict my recommendations but I am ready to see it again and to be more precise if you meet any difficulties.