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SPGC is true if it holds for all doubly short chord graphs

Zbigniew Lonc[1] and **Leszek S. Zaremba**[2]

Warsaw Uniwersity of Technology
Institute of Mathematics
Warsaw 00661, Poland

abstract

G is said to be a short chord graph if for each vertex v there exists an odd cycle on $2k+1$ ($k \geq 2$) vertices including v with all r chords adjacent to v such that if $k > 2$ then $r = 1$ and the only chord is triangular (short). G is said to be a doubly short chord graph if both G and its complement $\overline{G}$ is a short ch ord graph. We prove the SPGC is true if it holds for all doubly short chord graphs. This class ($\mathcal{G}$) is demonstrated to be nonempty. In addition, we show the SPGC is true for all graphs $G \notin \mathcal{G}$ if there are no graphs $G \in \mathcal{G}$ that are simultaneously Berge and p-critical. In our approach we use two known results, namely the Lovasz and Podberg characterizations of p-critical graphs. In the concluding section, we obtain two more results proved by a similar reasoning as in the first two theorems. They deal with so-called doubly chordally dense graphs (see Definition 3.1). The SPGC is again settled provided it holds for the above class of graphs.

1 Introduction

Claude Berge proposed to call a graph perfect if, for each of its induced subgraphs F, the chromatic number $F, \gamma(F)$, equals the size of a largest clique in $F, \omega(F)$. By an odd hole C_{2k+1} is meant a chordless cycle on $2k+1$ vertices, while $\overline{C}_{2k+1}$ stands for the complement of C_{2k+1}. The Strong Perfect Graph Conjecture (SPGC) says a graph is perfect if and only if none of its induced subgraphs is C_{2k+1} or $\overline{C}_{2k+1}$ (such graphs are called Berge graphs).

[1]Institute of Mathematics, Warsaw Uniwersity of Technology, Warsaw 00661, Poland

[2]Institute of Mathematics, Agric. & Pedag. University, SIEDLCE 08110, Poland. This work was partly done when the author was with Dept. of Math. & Stat., MUN, St. John's, Nfld, Canada A1C 5S7

While the forward implication ('only if') is obviously valid, the inverse implication is still unsettled. It is known that F-free Berge graphs (with no induced subgraph isomorphic to F) are perfect when F is, for example, $K_{1,3}$ "13—, the diamond"14—, the tetrahedron"17—, the bull"2— or the dart"16—.

"4— A number of interesting results on the SPGC of a different sort than the ones mentioned above have been published in the last years, see for example, "1,3-6,8-11,15—. For $k \geq 3$, let C_{2k+1}^v stand for a cycle on $2k+1$ vertices (including v) with exactly one chord, which is a triangular chord (joining two vertices whose distance on the cycle is two) adjacent to v. Let C_5^v mean a cycle on five vertices including v with one or two chords adjacent to v.

Definition 1.1 *We say $G = (V, E)$ a short chord graph (s.e.g. for short) if for each vertex $v \in V$ there exists a C_{2k+1}^v cycle, $k \geq 2$. G is said to be a doubly short chord graph if G and $\overline{G}$ are short chordgraphs (d.s.c.g. for short).*

The class of doubly short chord Berge graphs is nonempty. An example is shown in Figure 1. It is easy to check that for each vertex v of G there exists a C_5^v cycle and that there are no induced copies of C_5 and C_7. Since the complement $\overline{G}$ of G is isomorphic to G, the graph G is a d.s.c.g. and a Berge graph.

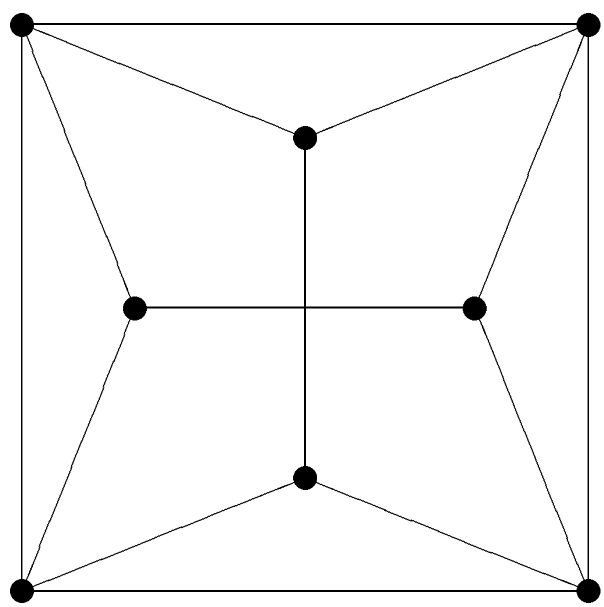

Figure 1.

To prove two theorems, we need hypotheses H_1, H_2 (clearly, H_2 implies H_1).

H_1. There are no graphs that are simultaneously Berge, p-critical and d.s.c .g.

H_2. Each Berge and d.s.c.g. is perfect.

We use H_1 to demonstrate that each Berge graph that is not d.s.c.g. is p erfect, while H_2 is sufficient to sette the SPGC. Independently of these two hypotheses, we make use of the Lovasz and Padberg characterizations of p-critical graphs. In the concluding section, we obtain two more results proved by a similar reasoning as in the first two theorems. They deal with so-called doubly chordally dense graphs (see Definition 3.1). The SPGC is again settled provided it holds for the above class of graphs.

2 Proofs of Theorems

Theorem 2.1 *Under hypothesis H_1, each Berge graph that is not a doubly short chord graph is perfect.*

Proof. Suppose the family $\mathcal{F}$ of Berge graphs that are not d.s.c.g. and not perfect is nonempty. Choose $G = (V, E) \in \mathcal{F}$ with the property that if $G' = (V', E') \in \mathcal{F}$ then $V \leq V'$. Observe G is p-critical (G is not perfect but each of its induced subgraph is perfect). Indeed, if not, G would contain a p-critical induced proper subgraph $G_A = (A, E_A)$ which, by hypothesis H_1, would belong to $\mathcal{F}$, a contradiction. Since G is not a d.s.c.g., either G or $\overline{G}$ contains a vertex v such that there is no cycle C^v_{2k+1}, $k \geq 2$. Without loosing a generality, assume G is such a graph. Since G is a Berge graph, $\omega(G) > 2$. Choose any $x \in V$ with $(x, v) \in E$. Since G is p-critical, $V - \{x\}$ is the union of $\alpha = \alpha(G)$ disjoint cliques $C_1, C_2, ...C_\alpha$ of size $\omega = \omega(G)$ covering $V \setminus \{x\}$ ($\alpha(G)$ denotes the size of a largest independent set in G). We may assume $v \in C_1$. By Padberg's theorem "12—, exactly ω cliques in G contain each vertex of G, particularly vertex v. Not counting C_1 , we thus have $\omega - 1$ cliques covering v, the latter implying (in view of $\omega > 2$) the existence of a vertex $b \neq x, b \notin C_1$, with $(v, b) \in E$. If we delete (v, b) from E, the resulting graph $G_{vb} = (V, E \setminus (v, b))$ is still not perfect for the Lovasz inequality "7— does not hold for G_{vb}. In fact, we have $V = 1 + \omega(G) \cdot \alpha(G) > \omega(G_{vb}) \cdot \alpha(G_{vb})$ because $\omega(G_{vb}) = \omega(G)$ and $\alpha(G_{vb}) = \alpha(G)$ the latter being a consequence of the fact that any independent set N in G_{vb} of size $\alpha + 1$ should contain the vertices x, v, b with $(x, v) \in E$, a contradiction. Namely, if either v or b did not belong to N, then N would be an independent set in G of size $\alpha + 1$, a contradiction.

Also, if $x \notin N$ then $N \subseteq \bigcup_{i=1}^{\alpha} C_i$ and because $N \cap C_i \leq 1$ we would have $\alpha + 1 = N \leq \alpha$.

Generalizing the last observation G_{vb} is not perfect, we delete from E all edges of the form (v, b), where $b \in \bigcup_{i=2}^{\alpha} C_i$, to conclude the resulting graf G_v is not perfect. Since v does not any longer belong to $\omega(G_v) = \omega(G)$ cliques, because of deletion of a number of edges containing v, G_v cannot be p-critical and consequently must contain a proper subgraph $G' = (V', E')$ that is p-critical with $\omega' = \omega(G') \geq 2$, $\alpha' = \alpha(G') \geq 2$. The set V' must contain vertex v and at least one vertex $b \in \bigcup_{i=2}^{\alpha} C_i$ because otherwise G' would be a perfect graph as a proper subset of the p-critical graph G.

Since $v \in G$' , by the mentioned above Padberg's theorem, v is contained in exactly ω' maximal cliques in G'. One of the cliques is $C_1 \cap V'$. By virtue of $\omega' \geq 2$, there must be another cligue containing v. Such a clique must contain a vertex not belonging to $C_1 \cap V'$. Because of deletion of all edges of the form (v, b), $b \in \bigcup_{i=2}^{\alpha} C_i$, the only candidate for such a vertex is x. This implies $\omega' = 2$, showing that $G' = C_{2k+1}$ in G_v, the latter being a partial subgraph of G. As a s ubgraph of G, $G' = (V', EV')$ is a cycle on $2k+1$ vertices, $k \geq 2$, with m chords of the form (v, b_i), $i = 1, ..., m$, which we deleted from G when constructing the graph G_v; denote this cycle with m chords by G".

If we show this is impossible for any value of m, we shall prove the theorem. The case $m = 1$ is easy because the only chord, being not a triangular chord by the choice of v, must "divide" G" (on at least seven vertices in this instance) on two cycles among which one has to be an odd cycle, a contradiction. Observe the assumption there is no a C_5^v cycle is needed to obtain a contradiction when $m = 2$ and G" is a cycle on five vertices. When G" is a cycle on $2k+1$ vertices with $k \geq 3$ and $m = 2$ (G" has two chords (v, b_1) and (v, b_2)), then we again arrive at a contradiction. Namely, if the both chords are triangular, then G" contains a chordless cycle C_{2l+1}, $l \geq 2$, whose t wo consequitive sides are (b_1, v) and (v, b_2). Otherwise, if exactly one of the two chords is triangular, G" must contain either an odd cycle with exatly one triangular chord, or an odd chordless cycle (both instances cannot occure due to our assumptions).

Finally, if none of the chords (v, b_1), (v, b_2) is triangular, then either at least one of them is a side of an odd chordless cycle (impossible), or both are consequitive sides of an odd chordless cycle C of length $2r + 1$ lying "in between" two other cycles of G". If $r = 1$ then G" must contain a cycle C_{2l+1}^v (impossible), while in the case $r \geq 2$ the cycle G" must contain the odd cycle C lying in between two even cycles (also impossible).

Suppose we have a method M to demonstrate an odd cycle on at least five vertices with m or less than m chords, all adjacent to vertex v, cannot exist in G (this is true for $m = 1$ and $m = 2$). Using M we show no odd cycle with $m + 1$ chords, all adjacent to v, exists in G. Indeed, at least one of $m \geq 3$ chords (v, b_i) "divides" G" into two cycles one of which, say C, must be odd of length at least five with at most m chords. The existence of such a cycle C contradicts, however, our method M.

□

Theorem 2.2 *Under hypothesis H_2 , the SPGC holds.*

Proof. If a Berge graph is a d.s.c.g., then it is perfect by hypothesis H_2 . Since H_2 implies H_1, Theorem 2.1 completes our proof. □

3 Conclusions

By a slight modification of our reasoning, one can obtain two more results.

Definition 3.1 *A graph G is said to be chordally dense if for each vertex x and each adjacent to it vertex v there exits a cycle on $2k+1,\ k \geq 2$, vertices with all $r \geq 1$ chords adjacent to v and such that x and v are two consequitive vertices of this cycle. G is said to be doubly chordally dense if both G and its complement $\overline{G}$ is chordally dense.*

An example of a doubly chordally dense graph is the graph presented in Figure 1.

H_1'. There are no graphs that are simultaneously Berge, doubly chordally dense and p-critical.

H_2'. Each Berge and doubly chordally dense graph is perfect.

Theorem 3.1 *Under hypothesis H_1', each Berge graph that is not doubly chordally dense is perfect.*

Proof. We argue similarly as in the first part of the proof of Theorem 2.1. In particular, we show that either G or $\overline{G}$ contains vertices v, x such that $(x, v) \in E$ or $(x, v) \notin E$ and there is no cycle in $G(\overline{G})$ on $2k+1,\ k \geq 2$, vertices containing x, v as consequitive vertices whose all $r \geq 1$ chords are adjacent to v. As previously, we delete from E all edges of the form (v, b), $b \in \bigcup_{i=2}^{\alpha} C_i$, to conclude the resulting graph is not perfect. Such a graph must then contain a proper p-critical subgraph G' containing x, v with $w' = w(G') = 2$, $\alpha' = \alpha(G') \geq 2$. As a subgraph of G, G' is an odd cycle whose all chords are adjacent to v, with x, v being two consequitive vertices of the cycle, a contradiction with the choice of x, v.

Theorem 3.2 *Under hypothesis H_2', the SPGC is true.*

The proof is the same as the one of Theorem 2.2.

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