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Two Generalized Trigonometric Fibonacci Sequences

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1 Introduction

This paper arises out of the Note 2 in [6]. First major generalization of the Fibonacci Sequence was formulated and studied by Horadam in his papers [2, 3, 4] He defined $\{W_n\} = \{W_n(a, b; p, q)\}$ given by

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2)$$

with initial values,

$$W_0 = a \quad and \quad W_1 = b.$$

Binet's formula and the exponential generating function $E(x)$ of $\{W_n'\}$ are respectively given by

$$W_n = \frac{l\alpha^n - m\beta^n}{2d}$$

and

$$E(x) = \frac{le^{\alpha x} - me^{\beta x}}{2d},$$

where α, β are the distinct roots of $x^2 - px + q = 0$ and l, m and d are given by

$$l = 2(b - a\beta), \;\; m = 2(b - a\alpha) \; and \; d = \alpha - \beta = \sqrt{p^2 - 49} \qquad (1.1)$$

It is clear that the Fibonacci Number Sequence $\{F_n\}$ is given by

$$F_n = W_n\{0,1;1,-1\}.$$

The exponential generating function $E_0(x)$ of $\{F_n\}$ is given by

$$E_0(x) = \frac{e^{\alpha_1 x} - e^{\beta_1 x}}{\sqrt{5}} \tag{1.2}$$

where α_1, β_1 are the distinct roots of $x^2 - x - 1 = 0$. In [1], Elmore uses (1.2) to formulate his generalization of F_n. He defines $E_n(x) = E_0^{(n)}(x)$, so that

$$E_n(x) = \frac{{\alpha_1}^n e^{\alpha_1 x} - \beta_1^n e^{\beta_1 x}}{\sqrt{5}} \quad (n \geq 1). \tag{1.3}$$

Note that $E_n(0) = F_n$.

Taking α, β as distinct roots of $x^2 - px + q = 0$ and using a similar process, we define

$$E^*{}_n(x) = \frac{\alpha^n e^{\alpha x} - \beta^n e^{\beta x}}{d}$$

where

$$d = \alpha - \beta$$

We easily see that

$$E^*{}_n(x) = pE^*{}_{n-1}(x) - qE^*{}_{n-2}(x).$$

Further

$$E^*{}_n(0) = F^*{}_n,$$

with

$$F^*{}_0 = 0, \quad F^*{}_1 = 1,$$

but

$$F_n^* = pF_{n-1}^* - qF_{n-2}^*.$$

Walton and Horadam [8] extended Elmore's results by obtaining another generalization of the Fibonacci Sequence from the generating function of $W_n(a,b;1,-1)$. They start with

$$H_0(x) = \frac{1}{2\sqrt{5}}\left[le^{\alpha}i^x - me^{\beta}i^x\right]$$

where l, m and d are as defined in (1.1) and then define

$$H_n(x) = H_0{}^{(n)}(x)$$

so that

$$H_n(x) = \frac{1}{2\sqrt{5}}\left[l{\alpha_i}^n e^{\alpha}{}_i{}^x - m{\beta_i}^n e^{\beta}{}_i{}^X\right] \quad (n \geq 1). \tag{1.4}$$

Note from (1.1) that for $a = 0$ and $b = 1$, (1.4) reduces to (1.3). Also observe that

$$H_n(0) = \frac{1}{2\sqrt{5}}\left[l{\alpha_i}^n - m{\beta_i}^n\right] = W_n(a, b, ; 1, -1),$$

so that for $a = 0, b = 1$,

$$H_n(0) = F_n.$$

We use Generalized Circular Functions to achieve two more generalizations of the Fibonacci Sequence.

2 Generalized circular functions

Let $r \geq 1$ be any fixed positive integer. The Generalized Circular Functions $N_{r,j}(t)$ and $M_{r,j}(t)$ are defined as

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \ldots, r-1 \tag{2.1}$$

and

$$M_{r,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, \ldots, r-1 \tag{2.2}$$

Observe that $N_{1,0}(t) = e^1$, $N_{2,0}(t) \cosh t$, $N_{2,1}(t) = \sinh t$ and $M_{1,0}(t) = e^{-t}$, $M_{2,0}(t) = \cos t$ and finally $M_{2,1}(t) = \sin t$. Mikusinski [5] studied these functions and proved some of their basic properties. The notation and some of the results used here are found in [7]. Differentiating (2.1) and (2.2) term by term, it is clear that

$$N_{r,j}^{(p)}(t) = \begin{cases} N_{r,j-p}(t), & 0 \leq p \leq j \\ N_{r,r+j-p}(t), & 0 \leq j < p \leq r \end{cases} \tag{2.3}$$

and

$$M_{r,j}^{(p)}(t) = \begin{cases} M_{r,j-p}(t), & 0 \leq p \leq j \\ -M_{r,r+j-p}(t), & 0 \leq j < p \leq r \end{cases} \tag{2.4}$$

In particular, note from (2.3) and (2.4) that

$$\left.\begin{array}{l} N^r{}_{r,j}(t) = N_{r,j}(t) \\ M^r{}_{r,j}(t) = -M_{r,j}(t) \end{array}\right\} \tag{2.5}$$

so that, in general,

$$\left.\begin{array}{l} N^{nr}{}_{r,j}(t) = N_{r,j}(t) \\ M^{nr}{}_{r,j}(t) = (-1)^n M_{r,j}(t) \end{array}\right\} \tag{2.6}$$

for $n \geq 1$.

3 Definition and preliminary results

Definition 3.1 *Let $r \geq 1$ be a fixed integer and $j = 0, 1, \ldots, r-1$. Let*

$$S_{j,0}(x) = \frac{1}{2d}\left[lN_{r,j}(\alpha^* x) - mN_{r,j}(\beta^* x),\right] \tag{3.1}$$

where α, β are distinct roots of $x^2 - px + q = 0$, $\alpha^* = \alpha^{1/r}$, $\beta^* = \beta^{1/r}$ and l, m and d are as defined in (1.1).

Note that

$$\alpha + \beta = p \quad \text{and} \quad \alpha\beta = q. \tag{3.2}$$

Now, define a sequence of Generalized Trigonometric Fibonacci Functions $\{S_{j,n}(x)\}$ as follows:

$$S_{j,1}(x) = S^r_{j,0}(x),$$

$$S_{j,2}(x) = S^{2r}_{j,0}(x),$$

and, in general,

$$S_{j,n}(x) = S^{(nr)}_{j,0}(x), \quad n \geq 1.$$

Then from (2.6) and (3.1) we get,

$$S_{j,1}(x) = \frac{1}{2d}\left[l\alpha N_{r,j}(\alpha^* x) - m\beta N_{r,j}(\beta^* x)\right],$$

$$S_{j,2}(x) = \frac{1}{2d}\left[l\alpha^2 N_{r,j}(\alpha^* x) - m\beta^2 N_{r,j}(\beta^* x)\right],$$

and, in general,

$$S_{j,n}(x) = \frac{1}{2d}\left[l\alpha^n N_{r,j}(\alpha^* x) - m\beta^n N_{r,j}(\beta^* x)\right], \quad n \geq 1. \tag{3.3}$$

Reduction to Fibonacci sequence

Observe from (2.1) that

$$N_{r,j}(0) = \begin{cases} 0, & j \neq 0 \\ 1, & j = 0. \end{cases}$$

Hence, it is clear that

$$S_{0,n}(x) = P_n(x)$$

where $P_n(x)$ is as defined in (3.4) of [6]. For $p = 1, q = -1, r = 1$ and $j = 0$, $S_{j,n}(x)$ reduces to Walton and Hordam's generalized Fibonacci Function $H_n(x)$ as defined in (4.1) of [8]. If, in addition to the above particular values, $a = 0$ and $b = 1$, then $S_{j,n}(x)$ reduces to Elmore's function $E_n(x)$. Finally, if in addition, $x = 0$ then $S_{j,n}9x)$ reduces to F_n.

For typographical convenience, we write $S_n(x)$ for $S_{j,n}(x)$ in the following sections.

Recurrence relation for $S_n(x)$.
$S_n(x)$ satisfies the following recurrence relation

$$S_n(x) = pS_{n-1}(x) - qS_{n-2}(x). \tag{3.4}$$

Proof:

$$\begin{aligned} \text{RHS} &= \tfrac{p}{2d}\left[l\alpha^{n-1}N_{r,j}(\alpha^* x) - m\beta^{n-1}N_{r,j}(\beta^* x)\right] \\ &= \tfrac{q}{2d}\left[l\alpha^{n-2}N_{r,j}(\alpha^* x) - m\beta^{n-2}N_{r,j}(\beta^* x)\right] \\ &= \tfrac{1}{2d}\left\{l\alpha^{n-2}N_{r,j}(\alpha^* x)[p\alpha - q] - m\beta^{n-2}N_{r,j}(\beta^* x)[p\beta - q]\right\}. \end{aligned}$$

As α, β are the distinct roots of $x^2 - px + q = 0$, we have $p\alpha - q = \alpha^2$ and $p\beta - q = \beta^2$, using which we easily get the result.

Some preliminary results

(A) Binets's Formula: $S_n(x)$ is given by

$$S_n(x) = \frac{1}{d}\left[(S_1(x) - \beta S_0(x))\alpha^n + (\alpha S_0(x) - S_1(x))\beta^n\right].$$

(B) Generating Function $S(t)$ and $S_n(x)$ is given by

$$S(t) = \sum_{n=0}^{\infty} S_n(x)t^n = \frac{S_0(x) - pS_0(x)t + S_1(x)t}{1 - pt + qt^2}$$

(C) Exponential Generating Function $E(t)$ for $S_n(x)$ is given by

$$E(t) = \sum_{n=0}^{\infty} \frac{S_n(x)t^n}{n!} = \frac{1}{d}\left[(S_1(x) - \beta S_0(x))e^{\alpha t} - (S_1(x) - \alpha S_0(x))e^{\beta t}\right].$$

(D)

$$\lim_{x\to\infty} \frac{S_{n+1}(x)}{S_n(x)} = \begin{cases} \beta & \text{if } \left|\frac{\alpha}{\beta}\right| < 1, \\ \alpha & \text{if } \left|\frac{\beta}{\alpha}\right| < 1. \end{cases}$$

(E)

$$\sum_{n=0}^{\infty} S_n(x) = \frac{S_{n+2}(x) - S_1(x) - (p-1)[S_{n+1}(x) - S_0(x)]}{p-q-1}$$

All the above results are easily proved by using (3.4) and by noting that α, β are the roots of $x^2 - px + q = 0$.

4 Various identities for $S_n(x)$

Again, for the sake of convenience, we write:

$$T_n(l, \alpha; x) \text{ for } l\alpha^n N_{r,j}(\alpha^* x)$$

and

$$T_n(m, \beta; x) \text{ for } m\beta^n N_{r,j}(\beta^* x).$$

Note that with this notation

$$S_n(x) = S_{j,n}(x) = \frac{1}{2d}\left[T_n(l, \alpha; x) - T_n(m, \beta; x)\right].$$

The following identities correspond to identities (4.2) to (4.8) in [6].

$$S_{n-1}(x)S_{n+1}(x) - S^2{}_n(x) = \frac{-T_n(l, \alpha; x)T_n(l, \beta; x)}{4q},$$

$$S_n(x)E^*{}_{n+1}(x) - qS_{n-1}(x)E^*(x) =$$

$$\frac{1}{2d}\left[e^{\alpha x}T_{n+r}(l, \alpha; x) - e^{\beta x}T_{n+r}(m, \beta; u)\right],$$

where $E_n^*(x)$ is Elmore's Function as defined in section 1.

$$S_n(u)E^*_{s+1}(v) - qS_{n-1}(u)E^*_s(v) = \frac{1}{2d}\left[e^{\alpha v}T_{n+s}(l, \alpha; x) - e^{\beta x}T_{n+s}(m, \beta; x)\right],$$

$$S_n^2(x) - qS_{n-1}^2(x) = \frac{1}{4d}\left[\alpha^{-1}T_n^2(l,\alpha;x) - \beta^{-1}T_n^2(m,\beta;x)\right],$$

$$S_{n+1}^2(x) - q^2S_{n-1}^2(x) = \frac{p}{4d}\left[T_n^2(l,\alpha;x) - T_n^2(m,\beta;x)\right],$$

$$S_{n-k+1}(x)S_{n+k+1}(x) - S_{n+1}^2(x) =$$

$$-\frac{1}{4d^2}\left[T_{n-k+1}(l,\alpha;x)T_{n-k+1}(m,\beta;x)\right](\alpha^k - \beta^k)^2,$$

$$S_n(x)S_{n+l+1}(x) - S_{n-k}S_{n+k+l+1}(x) =$$

$$\frac{1}{4d^2q^k}T_n(l,\alpha;x)T_n(m,\beta;x)(\alpha^k - \beta^k)(\alpha^{k+l+1} - \beta^{k+l+1}).$$

All the above identities are proved in a way similar to that in [6]. The proofs are therefore omitted. Observe that for $j = 0$, all the above identities are reduced to corresponding identities in [6], and in turn to the corresponding ones in [8] for $p = 1$, $q = -1$, $r = 1$ and $j = 0$. These, in turn, reduce to the corresponding identities for the Fibonacci Sequence for the particular values indicated in Section (3.2).

5 Associated generalized trigonometric Fibonacci sequence

The Associated Fibonacci Sequence, denoted by $\{F_n^*\}$, is defined by the following recurrence relation:

$$F_n^* = -F_{n-1}^* + F_{n-2}^*, \quad n \geq 2,$$

with the initial values

$$F_0^* = 0 \text{ and } F_1^* = -1.$$

The first few terms of this sequence are

$$0, -1, 1, -2, 3, -5, 8, -13.$$

It is easily proved by mathematical induction that

$$F_n^* = (-1)^n F_n.$$

Note that if the roots of $x^2+x-1=0$ are α_1^1 and β_1^1, then $\alpha_1^1=-\alpha_1$ and $\beta_1^1=-\beta_1$. Binet's formula for F_n^* is easily obtained as:

$$\begin{aligned} F_n^* &= (-1)^n F_n \\ &= \tfrac{(-1)^n\alpha_1^n-(-1)^n\beta_1^n}{\sqrt{5}} \\ &= \tfrac{\alpha_1^n-\beta_1^n}{\sqrt{5}}. \end{aligned}$$

Note that

$$F_n^* = W_n(0,-1;-1,-1).$$

One similarly defines the Associated Extended Fibonacci Sequence $\{W_n\}$ by the relation

$$W_n^* = (-1)^n W_n = \frac{l\alpha^{l^*} - m\beta^{l^*}}{2d}.$$

Then, it is easily proved that W_n^* satisfies the recurrence relation

$$W_n^* = -pW_{n-1}^* - qW_{n-2}^*, \quad n \geq 2$$

with

$$W_0^* = a \quad \text{and} \quad W_1^* = -b.$$

In other words $W_n^* = (a,-b;-p,q)$.

We now define the Associated Generalized Trigonometric Fibonacci Sequence $S_{j,n}^*(x)$. Let $r \geq 1$ be a fixed integer and $j = 0,1,\ldots,r-1$. Let,

$$S_{j,0}^* = \frac{1}{2d}\left[lM_{r,j}(\alpha^* x) - mM_{r,j}(\beta^* x)\right] \tag{5.1}$$

where α, β are distinct roots of $x^2-px+q=0$, $\alpha^* = \alpha^{1/r}$, $\beta^* = \beta^{1/r}$ and l, m and d are as defined in (1.1). Now define $\{S_{j,n}^*(x)\}$ as follows:

$$S_{j,1}^*(x) = S_{j,0}^{*(r)}(x),$$

$$S_{j,2}^*(x) = S_{j,0}^{*(2r)}(x),$$

and, in general,

$$S_{j,n}^*(x) = S_{j,0}^{*(nr)}(x), \quad n \geq 1.$$

Then, from (2.6) and (5.1), we get

$$S_{j,1}^*(x) = \frac{1}{2d}\left[-l\alpha M_{r,j}(\alpha^* x) + m\beta M_{r,j}(\beta^* x)\right],$$

$$S^*_{j,2}(x) = \frac{1}{2d}\left[-l\alpha^2 M_{r,j}(\alpha^* x) + m\beta^2 M_{r,j}(\beta^* x)\right],$$

and, in general,

$$S^*_{j,n}(x) = \frac{(-1)^n}{2d}\left[-l\alpha^n M_{r,j}(\alpha^* x) - m\beta^n M_{r,j}(\beta^* x)\right], \quad n \geq 1$$

If $p = 1, q = -1, j = 0$ and $r = 1$, we have, from (5.2),

$$\begin{aligned} S^*_{0,n}(x) &= \tfrac{(-1)^n}{2d}\left[l\alpha^n e^{-\alpha x} - m\beta^n e^{-\beta x}\right] \\ &= (-1)^n H_n(-x), \end{aligned}$$

where $H_n(x)$ is Walton and Horadam's Generalized Fibonacci Sequence as defined in (4.1) of [8].

Now Observe from (2.2) that

$$M_{r,j}(0) = \begin{cases} 0, & j \neq 0, \\ 1, & j = 0. \end{cases}$$

Hence, from (5.2), we get

$$\begin{aligned} S^*_{0,n}(0) &= \tfrac{(-1)^n}{2d}\left[l\alpha^n - m\beta^n\right] \\ &= \tfrac{1}{2d}\left[l\alpha^{l^n} - m\beta^{l^n}\right] \\ &= W^*_n. \end{aligned}$$

Finally, if $p = 1$, $q = -1$, $r = 1$ and $a = 0$, $b = 1$, then we have $l = m = 2$ and $d = \sqrt{5}$. Hence we get,

$$\begin{aligned} S^*_{0,n} &= \tfrac{(-1)^n(\alpha^n - \beta^n)}{\sqrt{5}} \\ &= F^*_n. \end{aligned}$$

6 Various results for $S^*_{j,n}(x)$

We now list below various identities for $S^*_{j,n}(x)$ which are parallel to those in sections (3.3), (3.4). Proofs also go parallel to those for $S_{j,n}(x)$ and hence are omitted. We write $S^*_n(x)$ for $S^*_{j,n}(x)$.

(1) $S^*_n(x)$ satisfies the following recurrence relation:

$$S^*_n(x) = -pS^*_{n-1} - qS^*_{n-2}.$$

In what follows α^l, β^l are the roots of $x^2+px+q=0$. Obviously $\alpha^l=-\alpha$ and $\beta^l=-\beta$.

(2) Binet's Formula for $S_n^*(x)$ is given by

$$S_n^*(x)=\frac{1}{d}\left[(\beta^l S_0^*(x)-S_1^*(x))\alpha^{l^n}+(S_1^*(x)-\alpha^l S_0^*(x))\beta^l\right].$$

(3) Generating Function $S^*(t)$ of $S_n^*(x)$ is given by

$$S^*(t)=\sum_{n=0}^{\infty} S_n^*(x)t^n=\frac{S_0^*(x)+pS_0^*(x)(t)+S_1^*(x)(t)}{1+pt+qt^2}$$

(4) Exponential Generating Function $E^*(t)$ for $S_n^*(x)$ is given by

$$E^*(t)=\sum_{n=0}^{\infty}\frac{S_n^*(x)t^n}{n!}=\frac{(-S_1^*(x)+\beta^l S_0^*(x))e^{\alpha^l t}+(S_1^*(x)-\alpha^l S_0^*(x))e^{\beta^l t}}{d}.$$

(5)

$$\lim_{x\to\infty}\frac{S_{n+1}^*(x)}{S_n^*(x)}=\begin{cases}\beta & \text{if } \left|\frac{\alpha}{\beta}\right|<1,\\ \alpha & \text{if } \left|\frac{\beta}{\alpha}\right|<1.\end{cases}$$

(6)

$$\sum_{n=0}^{\infty} S_n^*=\frac{(p+1)\left[-S_{m+1}^*(x)+S_0^*(x)\right]+S_1^*(x)-S_{m+2}^*(x)}{p+q+1}$$

In what follows, we write, for fixed integer $r\geq 1$, $T_n^*(l,\alpha;x)$ for $l\alpha^n M_{r,j}(\alpha^* x)$ and $T_n^*(m,\beta;x)$ for $m\beta^n M_{r,j}(\beta^* x)$.

(7)

$$S_{n-1}^*(x)S_{n+1}^*(x)-S_n^{*2}(x)=-\frac{1}{4q}T_n^*(l,\alpha;x)T_n^*(m\beta;x).$$

(8)

$$S_n^*(x)E_{n+1}^*(x)+qS_{n-1}^*(x)E_n^*(x)=$$
$$\frac{(-1)^n}{2d}\left[e^{\alpha x}T_{n+s}^*(l,\alpha;x)-e^{\beta x}T_{n+s}^*(m,\beta;x)\right].$$

(9)

$$S_n^*(u)E_{s+1}^*(v) + qS_{n-1}^*(u)E_s^*(v) =$$
$$\frac{(-1)^n}{2d}\left[e^{\alpha v}T_{n+s}^*(l,\alpha;u) - e^{\beta v}T_{n+s}^*(m,\beta;u\right].$$

(10)

$$S_n^{*2}(x) - qS_{n-1}^{*2}(x) = \frac{1}{4d}\left[\alpha^{-1}T_n^{*2}(1,\alpha;x) - \beta^{-1}T_n^*(m,\beta;x)\right].$$

(11)

$$S_n^{*2}(x) - q^2S_{n-1}^{*2}(x) = \frac{p}{4d}\left[T_n^{*2}(l,\alpha;x) - T_n^{*2}(m,\beta;x)\right].$$

(12)

$$S_{n-k+1}^*(x)S_{n+k+1}^*(x) - S_{n+1}^{*2}(x) =$$
$$\frac{1}{4d^2}\left[T_{n-k+1}^*(l,\alpha;x)T_{n-k+1}^*(m,\beta;x)\right](\alpha^k - \beta^k)^2.$$

(13)

$$S_n^*(x)S_{n+l+1}^*(x) - S_{n-k}^*(x)S_{n+k+l+1}^*(x) =$$
$$\frac{(-1)^{l+1}}{4d^2q^k}S_n^*(l,\alpha;x)S_n^*(m,\beta;x)(\alpha^k - \beta^k)(\alpha^{l+k+1} - \beta^{l+k+1}).$$

All the above identities are reducible to those for W_n^* and then, in turn , for F_n^*.

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$$k_n(x) = \sum_{v=0}^{\infty} \frac{(-1)^v x^{n+kv}}{(n+kv)!} \quad (k = 1, 2, \ldots; n = 0, 1, \ldots, k-1)^n.$$

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