Ulam Quarterly – Volume 3, Number 1, 1995

Notes on Contact Metric Manifolds

Ramesh Sharma

University of New Haven
Department of Mathematics
West Haven, CT 06516

1 Introduction

Let M denote a $(2n+1)$-dimensional contact metric manifold with contact form η, characteristic vector field ξ, the associated metric tensor g and the $(1,1)$-tensor field ϕ. We say M is K-contact if ξ is Killing. Following Blair [1] we denote the operators $(1/2)\mathcal{L}_\xi\phi$ and $R(.,\xi)\xi$ by h and I respectively; where $\mathcal{L}$ denotes Lie-derivative operator. The tensors h and I are known to be symmetric. We denote by τ the tensor metrically equivalent to the strain tensor $\mathcal{L}_\xi g$ of M along ξ, i.e. $g(\tau X, Y) = (\mathcal{L}_\xi g)(X,Y)$ for arbitrary smooth vector fields X, Y on M.

It is known that a contact metric manifold is K-contact if and only if $h = 0$. Recently the author [9] proved that a K-contact manifold cannot admit a second order symmetric parallel (covariant constant) tensor other than a constant multiple of the associated K-contact metric. This result has been generalized by the author [10] on a contact metric manifold the ξ-sectional curvature $K(\xi, X)$ non-vanishing and independent of the choice of X. On the other hand, a contact metric manifold satisfying $R(X,Y)\xi = 0$, is known (see Blair [2]) to be locally isometric to $E^{n+1} \times S^n$ (4) and is known (Blair and Patnaik [3]) to admit the second order symmetric parallel tensor $h - \eta \otimes \xi$. Question, therefore arises where there are contact metric manifolds for which one of the symmetric tensors h, I and τ is parallel. We answer it a little more generally by proving

Theorem 1.1 *Let M be a contact metric manifold. Then*

(i) if h is a Codazzi (in particular, parallel) tensor then $h = 0$, i.e. M is K-contact.

(ii) if τ is Codazzi (in particular, parallel) then $\tau = 0$, i.e. M is K-contact.

(iii) if I is parallel then $I = 0$, i.e. the sectional curvatures of all plane sections containing ξ vanish.

Remark 1.1 *Part (iii) of the above theorem deserves special attention because I is the operator that measures the sectional curvatures of plane sections containing ξ and the theorem says that if I is parallel then $I = 0$. However the question of classifying locally symmetric (i.e. parallel Riemann tensor) contact metric manifolds is still open, except in dimensions 3 and 5 (see [5, 6]).*

Now if A is a Codazzi tensor on a Riemannian manifold with Riemannian connection ∇, i.e. $(\nabla_X A)Y = (\nabla_Y A)X$ then $(\mathrm{div}\, A)X = X(\mathrm{Tr}\, A)$. We would like to examine the converse situation for $A = h$ on a contact metric manifold M. but $\mathrm{Tr}\, h = 0$ on M and hence we need to examine just the condition $\mathrm{div}\, h = 0$. We answer it in dimension 3 proving

Theorem 1.2 *For a contact metric 3-manifold M the condition that $div\, h = 0$ is equivalent to the condition that ξ is a eigenvector of the Ricci operator Q.*

Remark 1.2 *Note that the condition $Q\xi$ = a function multiple of ξ, is equivalent on a contact metric manifold, to $div\, \tau$ = a function multiple of η. An example of a contact metric manifold satisfying this condition is given [4] by $\mathbb{R}^3$ with contact structure $\eta = 1/2(\cos x^3 dx^1 + \sin x^3 dx^2)$ and the associated metric $g_{ij} = (1/4)\delta_{ij}$. Since η is invariant by the translations in the coordinate directions by 2π, the torus T^3 is a compact manifold also carrying this structure. For this contact metric structure, $div\, \eta = -4\eta$.*

Remark 1.3 *In view of the identities $Ric\,(X, \xi) = 2\eta(X) + (div\,(h\phi))(X)$ and $\nabla_X \xi = -\phi X - \phi h X$ for a contact metric 3-manifold, it follows that the condition that ξ is a eigenvector of the Ricci operator, is equivalent to the condition that ξ be a eigenvector of the Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$. Contact metric 3-manifolds with ξ as an eigenvector of Δ have been classified by H. Chen [7] and are either Sasakian or locally isometric to a Lie-group with left-invariant metric. It is also important to note that the condition that ξ is a eigenvector of Ricci operator, is invariant under the D-homothetic deformation of the contact metric structure, defined as (see [11, 4]):*

$$\bar{\eta} = \alpha\eta, \;\; \bar{\xi} = \frac{1}{\alpha}\xi, \;\; \bar{\phi} = \phi, \;\; \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$$

for a positive constant α. $(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is again a contact metric structure.

2 Preliminaries

A $(2n+1)$-dimensional C^∞-manifold M is said to be a contact manifold if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M.

For a contact form η there exists a vector field ξ such that $\eta(\xi) = 1$ and $(d\eta)(\xi, X) = 0$ for any vector field X. A Riemannian metric g is said to be an associated metric if there exists a $(1,1)$ tensor field ϕ such that $(d\eta)(X,Y) = g(X,\phi Y), \eta(X) = g(\xi, X)$ and $\phi^2 = -I + \eta \otimes \xi$. The manifold M with the structure (η, ξ, ϕ, g) is called a contact metric manifold. The tensor field $h = (1/2)\mathcal{L}_\xi \phi$ is symmetric, traceless and satisfies $h\xi = 0$ and $h\phi = -\phi h$. If λ is an eigenvalue of h with eigenvector E then $-\lambda$ is also an eigenvalue with eigenvector ϕE. For details we refer to [1]. On a contact metric manifold,

$$\nabla_\xi \phi = 0 \tag{2.1}$$

$$\nabla_X \xi = \phi X - \phi h X (\Rightarrow \nabla_\xi \xi = 0 \text{ and } \operatorname{div} \xi = 0) \tag{2.2}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2g(h\phi X, Y) \tag{2.3}$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X \tag{2.4}$$

$$(\nabla_\xi h)X = \phi\{X - h^2 X + R(\xi, X)\xi\} \tag{2.5}$$

$$\operatorname{div} \phi = -2n\eta \tag{2.6}$$

Formulas (2.2) and (2.4) occur in [2], (2.5) in [3] and (2.6) in [8]. From (2.4) it follows that the sum of the sectional curvatures $K(\xi, X)$ and $K(\xi, \phi X)$ for a unit vector X orthogonal to ξ is $= 2(I - |hX|^2)$. Thus, if $K(\xi, X) \geq 0$ for any vector X orthogonal to ξ then $K(\xi, X) \leq 2$. We also note that, if $K(\xi, X) \geq 1$ then $K(\xi, X) = 1$. A contact metric manifold is K-contact (i.e. ξ is Killing) if and only if $K(\xi, X) = 1$ for any X orthogonal to ξ.

3 Proofs of the results

Proof of Theorem 1.1 Substituting ξ for Y in the hypothesis $(\nabla_X h)Y = (\nabla_Y h)X$ and using (2.2) we find

$$h\phi X + h\phi h X = (\nabla_\xi h)X. \tag{3.1}$$

Using $h\phi = -\phi h$ and (2.5) in the above equation gives

$$R(\xi, X)\xi = \phi^2 X - h^2 X \tag{3.2}$$

because $\eta(R(\xi, X)\xi) = 0$. Use of (3.1) in (2.4) shows $h^2 = 0$. Therefore, $|h|^2 = \operatorname{Tr} h^2 = 0$ and hence $h = 0$, proving (i). To prove (ii), by hypothesis and (2.3) we have

$$(\nabla_X(h\phi))Y = (\nabla_Y(h\phi))X$$

Substituting ξ for Y and using (2.1), (2.2) and (2.5) gives

$$h\phi(\phi + h)X = (h^2 + \phi^2)X + \phi R(\xi, \phi X)\xi$$

Applying ϕ to both sides and using $h\phi = -h\phi$ and $h\xi = 0$ we get

$$R(\xi, \phi X)\xi = \phi(hX - X) + (\phi - I)h^2 X \tag{3.3}$$

Replacing X by ϕX gives

$$R(\xi, X)\xi = h(h + h\phi - I)X + \phi^2 X \tag{3.4}$$

The use of (3.2) and (3.3) in (2.4) yields $h^2\phi X = 0$. Replacing X by ϕX gives $h^2 = 0$. As h is symmetric, it follows that $h = 0$. This proves (ii). For (iii) we first differentiate (2.4) along ξ (understanding that $IX = R(X, \xi)\xi$) and using the fact $\nabla_\xi \phi = 0$, we obtain

$$\nabla_\eta h^2 = 0 \tag{3.5}$$

That is,

$$h(\nabla_\xi h) + (\nabla_\xi h)h = 0$$

Differentiating it covariantly along ξ gives

$$2(\nabla_\xi h)^2 + h(\nabla_\xi \nabla_\xi h) + (\nabla_\xi \nabla_\xi h)h = 0. \tag{3.6}$$

Now differentiating (2.5) along ξ covariantly gives

$$\nabla_\xi \nabla_\xi h = 0 \tag{3.7}$$

Using (3.6) in (3.5) gives $(\nabla_\xi h)^2 = 0$. Hence

$$\nabla_\xi h = 0 \tag{3.8}$$

Then using (3.7) in (2.5) yields

$$\phi(X - h^2 X + X - IX) = 0$$

Operating by ϕ gives

$$IX = -h^2 X + X - \eta(X)\xi. \tag{3.9}$$

Next, differentiating (3.8) covariantly along Y, using the hypothesis $\nabla I = 0$ and then substituting ξ for X gives

$$h^2\phi Y - h^3\phi Y - \phi Y + h\phi Y = 0 \tag{3.10}$$

At this stage, first replacing Y by ϕY in (3.9) gives

$$-h^2Y + h^3Y + Y - \eta(Y)\xi - hY = 0. \tag{3.11}$$

Secondly, operating (3.10) by ϕ gives

$$-h^2Y - h^3Y + Y - \eta(Y)\xi + hY = 0 \tag{3.12}$$

Adding (3.10) to (3.11) shows

$$h^2 + \phi^2 = 0 \tag{3.13}$$

Consequently (3.8) and (3.12) imply $I = 0$, ending the proof.

Proof of Theorem 1.2 From (2.2) it follows for a $(2n+1)$-dimensional contact metric manifold M that

$$(\nabla_X h)\xi = h(\phi - h\phi)X \tag{3.14}$$

Taking (e_i) as a local orthonormal basis, putting $X = e_i$ in (3.13), taking inner product with e_i and summing over i we find

$$(\text{div } h)\xi = 0 \tag{3.15}$$

as Tr $(h\phi) =$ Tr $(h^2\phi) = 0$. If $h = 0$ (i.e. M is K-contact) then (i) and (ii) are obviously true. So let $h \neq 0$ in some neighborhood of M and now let $\dim M = 3$, i.e. $n = 1$. We can choose a local ϕ-basis $(\xi, E, \phi E)$ such that $hE = \lambda E$ and $h\phi E = -\lambda\phi E$. From (2.3) and the definition of τ, we have $\tau = 2h\phi$. Hence

$$(\text{div } \tau)Y = 2[(\text{div } h)\phi Y + \lambda\{2g((\nabla_E\phi)Y, E) - (\text{div } \phi)Y\}].$$

Using (2.6) gives

$$(\text{div } \tau)Y = 2(\text{div } h)\phi Y + 2\lambda[\eta(Y) + 2g((\nabla_E\phi)Y, E)]. \tag{3.16}$$

To prove (i) implies (ii), assume div $h = 0$. Then (3.15) gives

$$(\text{div } \tau)Y = 2\lambda[g((\nabla_E\phi)Y, E) + \eta(Y)].$$

Taking $Y = E$, we find

$$(\text{div } \tau)E = 4\lambda g((\nabla_E\phi)E, E) = 0.$$

Thus $(\text{div } \tau)X = 0$ for any X orthogonal to ξ. Hence div $\tau = f\eta$ for some scalar field f. Conversely, to prove (ii) implies (i), assume div $\tau = f\eta$. Then

(3.15) gives

$$(f - 2\lambda)\eta(Y) = 2(\text{div } h)\phi Y + 4\lambda g((\nabla_E \phi)Y, E).$$

Replacing Y by ϕE,

$$(\text{div } h)E = 2\lambda g((\nabla_E \phi)\phi E, E) = 2\lambda[-g(\nabla_E E, E) + g(\nabla_E \phi E, \phi E)] = 0,$$

since E and ϕE are unit vector fields. As E is any unit eigenvector orthogonal to ξ, of h, we obtain $(\text{div } h)X = 0$ for any X orthogonal to ξ. Taking into account this and (3.14) we conclude $\text{div } h = 0$. This completes the proof.

Acknowledgment: The author is grateful to Professor David E. Blair for his going through the manuscript.

References

[1] D.E. Blair. *Contact manifolds in Riemannian geometry, Lecture notes in Mathematics 509.* Springer-Verlag, Berlin-Heidelberg, New York, 1976.

[2] D.E. Blair. Two remarks on contact metric structures. *Tôhoku Math J.*, 29:319–324, 1977.

[3] D.E. Blair and J.N. Patnaik. Contact manifolds with characteristic vector field annihilated by the curvature. *Bull. Inst. Math. Acad. Sinica*, 9:533–545, 1981.

[4] D.E. Blair and R. Sharma. Generalization of myers' theorem on a contact manifold. *Illinois J. Math.*, 34:837–844, 1990.

[5] D.E. Blair and R. Sharma. Three dimensional locally symmetric contact metric manifold. *Bolletino U.M.I.*, 7(4-A):385–390, 1990.

[6] D.E. Blair and J.M. Sierra. 5-dimensional locally symmetric contact metric manifolds. preprint.

[7] H Chen. Contact metric 3-manifolds with characteristic vector as an eigenvector of the laplacian. private communication.

[8] Z. Olszak. On contact metric manifolds. *Tôhoku Math J.*, 31:247–253, 1979.

[9] R. Sharma. Second order parallel tensors on contact manifolds. *Algebras, Groups and Geometries*, 7:145–152, 1990.

[10] R. Sharma. Second order parallel tensors on contact manifolds ii. *C.R. Math. Rep. Acad. Sci. Canada*, 13:259–264, 1991.

[11] S. Tanno. The topology of contact riemannian manifolds. *Illinois J. Math*, 12:700–717, 1968.

This electronic publication and its contents are ©copyright 1995 by Ulam Quarterly. Permission is hereby granted to give away the journal and it contents, but no one may "own" it. Any and all financial interest is hereby assigned to the acknowledged authors of the individual texts. This notification must accompany all distribution of Ulam Quarterly.