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A C^∞ Spline Construction

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Abstract

We consider the problem of determining a smooth spline through a set of points in the plane determined using a standard partition of unity of construction.

1. Introduction

Theorem 1 *Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers and $d_0, d_1, d_2, \ldots$ another sequence of real numbers defined by the formula*

$$\begin{aligned} d_0 &= \text{arbitrary, and} \\ d_i &= (a_{i+1} - a_{i-1})/2 \text{ for } i > 0 \ . \end{aligned} \tag{1}$$

(This is the 3-point midpoint formula for approximating the derivative at (i, a_i).) Given $\epsilon > 0$, then there exists a function $f(x) \in C^\infty[0, \infty)$ satisfying

1. *$f(i) = a_i$ for $i \geq 0$;*
2. *$f'(i) = d_i$ for $i \geq 0$;*
3. *$\sup_x |f(x) - l(x)| < \epsilon$ where $l(x)$ is the linear spline satisfying*
   - *(a) $l(i) = a_i$ for $i \geq 0$,*
   - *(b) $l(x)$ is linear on $[i, i+1]$ for $i \geq 0$;*

   and,

4. *On each interval $[i, i+1]$ for $i \geq 0$, $f(x)$ has at most one inflection point. More precisely, if m_i denotes the slope of $l(x)$ on $[i, i+1]$ then, $f(x)$ has no inflection points on $[i, i+1]$ provided $(d_i - m_i)(d_{i+1} - m_i) < 0$. On the other hand, if $(d_i - m_i)(d_{i+1} - m_i) \geq 0$, then $f(x)$ has a unique inflection point in $[i, i+1]$ at the midpoint $x = i + \frac{1}{2}$.*

Proof. First we review the existence of certain C^∞ functions. Let

$$r(x) = \begin{cases} e^{-\frac{1}{x^2} - \frac{1}{(x-1)^2}} & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

It is well known that $r(x)$ is infinitely differentiable at all points $x \in \mathbb{R}$. Let

$$t(x) = \left(\int_{-\infty}^{\infty} r(x)\, dx \right)^{-1} \int_{-\infty}^{x} r(u)\, du . \tag{3}$$

Then $t(x)$ is C^∞, $t(x) = 0$ for $x \leq 0$, $t(x) = 1$ for $x \geq 1$ and $t'(x) = r(x) > 0$ if $0 < x < 1$. Note that derivatives of all orders (except zeroth) of $t(x)$ vanish at $x = 0$ and $x = 1$. All functions to be constructed in the proof of Theorem 1 will be built out of $t(x)$ by means of formulae. This will imply that all functions constructed below will depend continuously on the parameters of the construction.

Using affine transformations in x and in y ($= t(x)$), we see easily that for every interval $[a, b]$ and every pair of values $\alpha, \beta \in \mathbb{R}$ there exists a C^∞ function on $\mathbb{R}$, $j_\xi(x)$, $\xi = (a, b, \alpha, \beta)$ such that

$$\begin{aligned} j_\xi(x) &= \alpha \text{ for } x \leq a \\ j_\xi(x) &= \beta \text{ for } x \geq b \text{ and} \\ j'_\xi(x) &\neq 0 \text{ for } x \in (a, b) . \end{aligned} \tag{4}$$

Similarly, given $\zeta = (a, b, c, \alpha, \beta, \gamma)$, $a < b < c$, there exists a smooth function $k_\zeta(x)$ on $\mathbb{R}$ with

$$\begin{aligned} k_\zeta(x) &= \alpha \text{ for } x \leq a \\ k_\zeta(x) &= \gamma \text{ for } x \geq c \text{ and} \\ k_\zeta(b) &= \beta . \end{aligned} \tag{5}$$

We can simply set

$$\begin{aligned} k_\zeta(x) &= j_{(a,b,\alpha,\beta)}(x) \text{ for } x \leq b \\ k_\zeta(x) &= j_{(b,c,\beta,\gamma)}(x) \text{ for } x \geq b . \end{aligned} \tag{6}$$

Clearly $k_\zeta(x)$ has the required properties.

Note that if ζ is as above and a, b, c, α, γ are kept fixed, then

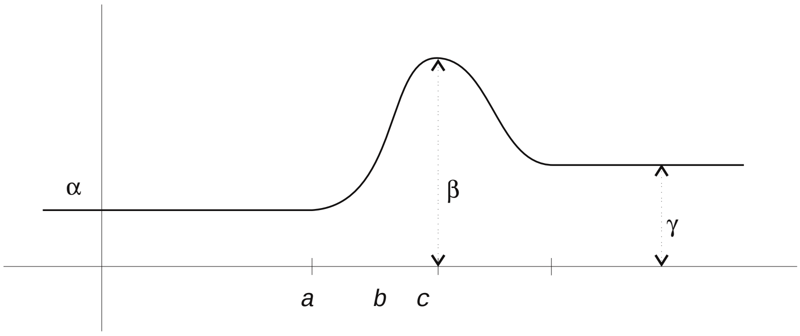

Fig. 1: $y = k_\zeta(x)$.

$$\lim_{\beta\to\infty} \int_a^c k_\zeta(x)\, dx = \infty \ . \tag{7}$$

This observation will be used repeatedly below.

2. Construction

We are now ready to start the construction. We will work on one interval $[i, i+1]$ at a time and construct $f_i = f|[i, i+1]$ satisfying in addition to properties (1), (2), (3), (4) listed in the theorem, the property

(5) $f_i^{(k)}(i) = f_i^{(k)}(i+1) = 0$ for all $k \geq 2$.

Condition (5) will ensure that the f_i's fit together smoothly. Now, since $l(x)$ is a known function, it is enough to construct $\phi_i = f_i - l_i$, where $l_i = l|[i, i+1]$. The properties that ϕ_i must satisfy are then

(1') $\phi_i(i) = \phi_i(i+1) = 0$

(2') $\phi_i'(i) = e_i$, $\phi_i'(i+1) = e_{i+1}$, where $e_i = d_i - m_i$ is prescribed.

(3') $|\phi_i(x)| < \epsilon$ for every $x \in [i, i+1]$

(4') If $e_i \cdot e_{i+1} \geq 0$, then $\phi_i|(i, i+1)$ has exactly one inflection point located at $i + \frac{1}{2}$. If $e_i \cdot e_{i+1} < 0$, then $\phi_i|(i, i+1)$ has no inflection point.

Clearly it suffices to describe the construction of $\phi = \phi_0$. The same construction will work for every interval $[i, i+1]$. The idea is to guess the shape of the graph of the second derivative of ϕ and integrate twice.

Case 1. Consider $e_0 \cdot e_1 < 0$. If necessary, multiply all functions by -1, to be able to assume that $e_0 > 0$, $e_1 < 0$. The functions $y = \phi(x)$, $y = \phi'(x) = \rho(x)$ and $y = \phi''(x) = \psi(x)$ have graphs that look like the following.

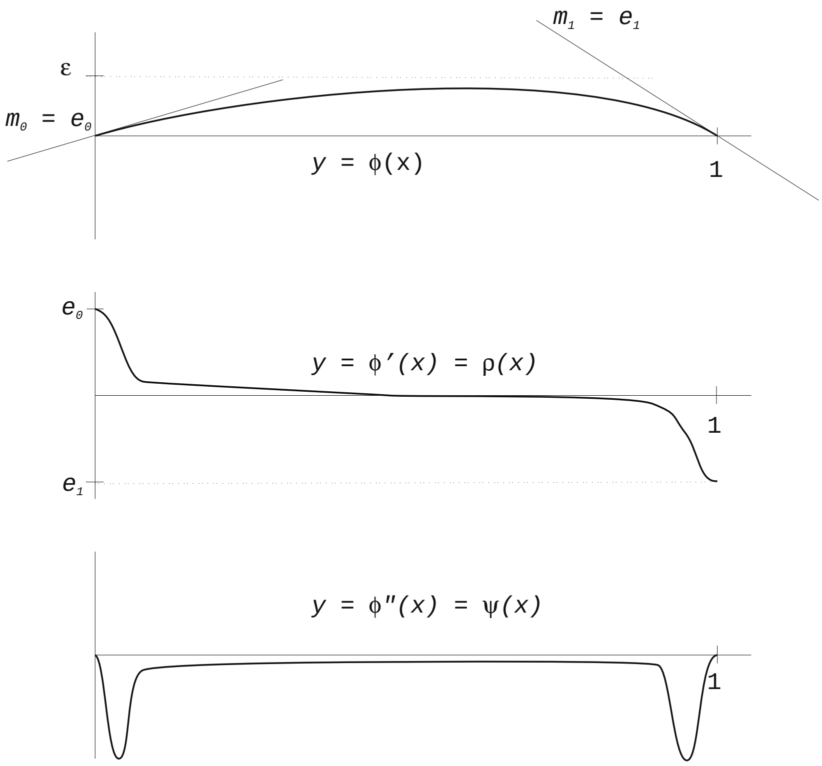

Fig. 2: The function ϕ and its first 2 derivatives.

The functions $\psi(x)$ and $\rho(x)$ must satisfy

$$\int_0^1 \psi(x)\,dx = e_1 - e_0 \tag{8}$$

$$\int_0^1 \rho(x)\,dx = 0 \tag{9}$$

$$\left|\int_0^x \rho(u)\,du\right| \leq \epsilon \text{ forall } x \in [0,1]\ . \tag{10}$$

If this is the case, then

$$\rho(x) = e_0 + \int_0^x \psi(u)\,du\ , \tag{11}$$

$$\phi(x) = \int_0^x \rho(u)\,du \tag{12}$$

and ϕ satisfies the required properties.

We will take the widths of the two spikes, the values of ψ at the local minima and the value near $x = \frac{1}{2}$ as parameters and adjust them so that (8), (9) and (10) are satisfied. Thus, let η_0, η_1, μ be three (small) positive numbers and let h_0 and h_1 be two negative numbers. Define

$$\psi(x) = \begin{cases} k_{(0,\frac{\eta_0}{2},\eta_0,0,h_0,-\mu)}(x) & \text{for } 0 \le x \le \frac{1}{2} \\ k_{(1-\eta_1,1-\frac{\eta_1}{2},1,-\mu,h_1,0)}(x) & \text{for } \frac{1}{2} \le x \le 1 \end{cases} . \tag{13}$$

Thus η_0 and η_1 measure the widths of spikes, $-h_0$ and $-h_1$ are depths and μ is the depth of the horizontal portion of the graph. Given η_0, η_1, μ (provided μ is such that $0 < \mu \le |e_1 - e_0|$) we can choose $h_0 = h_1 = h$ so that

$$\int_0^1 \psi(x)\, dx = e_1 - e_0 < 0 \tag{14}$$

This h is actually unique for a fixed choice of η_0, η_1, μ. Since e_0 and e_1 are fixed it is also obvious that (10) holds for all choices of η_0, η_1, μ which are sufficiently small. This is because η_0, η_1 and μ determine how closely the graph of $\rho(x)$ hugs the lines $x = 0$, $x = 1$ and $y = 0$. We still need to make sure that (9) holds. Suppose this is not the case for some choice of η_0 and η_1 and, for example,

$$\int_0^1 \rho(x)\, dx > 0 \ . \tag{15}$$

Let $\rho_+ = \frac{1}{2}(\rho + |\rho|)$, $\rho_- = \frac{1}{2}(|\rho| - \rho)$ be the positive and negative parts of ρ respectively. Then

$$\int_0^1 \rho_+(x)\, dx > \int_0^1 \rho_-(x)\, dx \tag{16}$$

by (15). First choose μ so that

$$0 < \mu \ll \int_0^1 \rho_-(x)\, dx \ . \tag{17}$$

This makes the shaded area in Figure 3 negligible.

Now let η_0 approach zero keeping η_1 fixed. Clearly for η_0 sufficiently small $A_1 \ll A_2$ so that

$$\int_0^1 \rho(x)\, dx < 0 \ . \tag{18}$$

Therefore for some intermediate value of η_0

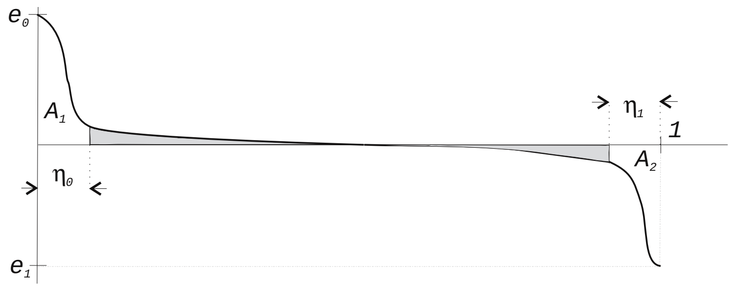

Fig. 3: The function ρ.

$$\int_0^1 \rho(x)\,dx = 0\ . \tag{19}$$

This finishes the construction in Case 1.

Case 2. Consider$e_0 \cdot e_1 \geq 0$. This is actually slightly easier. Suppose $e_0 \cdot e_1 > 0$. The case when one of the slopes is equal to 0 requires a modified argument and will be treated later. First consider the graphs.

In order for $\phi(x)$ to have all of the required properties, we must have conditions (8), (9) and (10) satisfied. We again try to construct ψ. Let η_0, η_1 and η be parameters controlling the width of the spikes near 0, 1 and $\frac{1}{2}$ respectively. Only one parameter is needed in the middle since we require that

$$\eta(\frac{1}{2} - t) = -\eta(\frac{1}{2} + t) \tag{20}$$

for, say, $|t| < \frac{1}{4}$. Similarly let $\mu = |\psi(x)|$ for x near $\frac{1}{4}$ or for x near $\frac{3}{4}$. We need three more parameters that give the heights of spikes: h_0, h_1, and h. Observe that because of symmetry,

$$\int_0^1 \psi(u)\,du \tag{21}$$

is independent of h. Given η_0, η_1, η, h we can find h_0 and h_1 so that (8) holds. This is obvious. We can now vary h to achieve

$$\int_0^1 \rho_+(x)\,dx = \int_0^1 \rho_-(x)\,dx \tag{22}$$