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All Bundles in $W^3_{2,8}$ Have Infinitely Many Maximal Subbundles

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1. Introduction

Let X be a complete non-singular irreducible curve over an algebraicaly closed field of characteristic zero. For a vector bundle E of rank two over X define

$$s(E) = \mathrm{deg(E)} - 2 \max \mathrm{deg(L)},$$

where the maximum is taken over all line subbundles L of E. A line bundle L of E of maximal degree will be called a maximal line subbundle. Let $M(E)$ be the sub-scheme of $Pic(X)$ formed by the maximal line subbundles of E. Maruyama [M] proved the following results:

1. If $s(E)$ =g, then dim $M(E) = 1$
2. For a general bundle E with $s(E) \leq g - 1$ there are only finitely many maximal subbundles.

Maruyama conjectured, that if E is not of the form $L \oplus L$ and $s(E) \leq g - 1$, the number of maximal subbundles is finite. Lange and Narasimhan [LN] showed that the conjecture is not generally true. For $s(E) = 2$ and $g \geq 3$ they determined completely the curves and bundles E for which $M(E)$ is not finite. For $s(E) = 3$ and $g \geq 4$ they showed that there is a curve X of genus g and a vector bundle E on X with dim $M(E) = 1$. They also construct examples with dim $M(E) = 1$ for arbitrary g large enough compared with $s(E)$.

2. Discussion

Let $W^r_{n,d} = \{E : \ h^0(E) \geq r+1\}$ be a sub-variety of moduli space of stable vector bundles on X of degree d rank n consisting of bundles E such that $h^0(E) \geq r+1$.

Theorem: *Let X be a general curve of genus 5. Then all bundles in*

$$W^3_{2,8} = \{E : \ h^0(E) \geq 4\},$$

have infinitely many maximal subbundles.

Proof: In [T, Theorem 1] Teixidor showed that for a general curve of degree 5 $W^3_{2,8}$, is not empty. Let E be in $W^3_{2,8}$. By stability of E for any subbundle L we have deg $L < 4$. Assume that E has a subbundle of degree 3, denoted as L^3. Then we could write E as an extension

$$0 \to L^3 \to E \to L^5 \to 0,$$

where L^5 is a quotient bundle of degree 5.

Define $\rho = g - (r+1)(g-d+r)$, where g is a genus of a curve X, d and r are integers such that $d > 0, r \geq 0$. By Dimension Theorem for line bundles [ACGH, pg 214] for $\rho < 0$ there is no bundles having $h^0(L) \geq r+1$. Therefore $h^0(L^3) \leq 1$ and $h^0(L^5) \leq 2$. But then $h^0(E) \leq 3$ which contradicts the assumption that E has $h^0(E) \geq 4$.

This implies that degree of a maximal subbundle of E must be less then 3. Since $h^0(E) \geq 4$ and deg $E = 8$ there is an infinite family of line subbundles of degree 2 of E. Therefore E has infinitely many maximal subbundles.

Remark 1. Note that in this case $d \equiv g - 1 mod 2$ hence by Corollary 3.2(Maruyama) in [LN] there is an open dense set V of moduli space such that every $E \in V$ admits only a finite number of maximal subbundles. Since $W^3_{2,8}$ contains only bundles with an infinite number of maximal subbundles the complement of V is not empty.

Remark 2. Since in our case $s(E) = 4$, $W^3_{2,8}$ gives new examples of bundles with an infinite number of maximal subbundles.

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